Unitarily manipulating in time and space a Gaussian wave-packet motional state of a single atom in a quadratic potential field

Xijia Miao
Somerville, Massachusetts
Date: July 2007

Abstract

The paper first discusses theoretically the off-resonance selective excitation method that is dependent on the atomic internal states and used to generate approximately a standard coherent state of harmonic oscillator. The coherent average method then is proposed to construct the state-selective trigger pulse. A state-selective trigger pulse can keep Gaussian shape unchanged but change in an internal-state-dependent form the center-of-mass position and/or momentum of an atomic Gaussian wave-packet motional state. A Gaussian wave-packet state is one of the simplest wave-packet states that can be easily manipulated and controlled in time and space. The paper also investigates how to manipulate in time and space an atomic Gaussian wave-packet motional state by a generalized quadratic potential field. A general quadratic Hamiltonian can affect not only the center-of-mass position and momentum but also the complex linewidth of a Gaussian wave-packet motional state while keep Gaussian shape of the motional state unchanged. It is shown that generally quadratic terms of a quadratic Hamiltonian can control directly the complex linewidth, while linear terms of a quadratic Hamiltonian can affect only the center-of-mass position and momentum of a Gaussian wave-packet motional state.

1. Introduction

In the paper [1] a particle picture has been used to describe intuitively how the halting-qubit atom evolves in the state-locking pulse field and how to construct a quantum control process to simulate the reversible and unitary halting protocol that is insensitive to its input state, although the quantum mechanical wave-packet picture has also used extensively in that paper. A particle picture is particularly intuitive to describe the decelerating and accelerating processes of a free atom and the elastic collision process for an atom bouncing off a hard potential wall. The correctness of a particle picture is based on the fact that a particle picture is very close to a wave-packet picture in quantum mechanics [2]. However, a quantitative and exact calculation in quantum mechanics for the time evolution process of a quantum system such as an atom does not use the classical particle picture. One must use wave-packet states or more generally quantum states to calculate the time evolution process of the atom in quantum mechanics. The wave-packet states were used by Schrödinger, Dirac, and others to describe the quantum mechanical behavior of a particle date back to the early time of the wave mechanics. A very familiar example is that a free

particle such as an atom may be described exactly by a Gaussian wave packet state in quantum mechanics [2]. Wave-packet states and especially Gaussian wave-packet states have been used frequently to describe the quantum collision and scattering processes in the atomic and molecular systems [2, 3, 4]. It can be seen in Refs. [5, 6, 7] that there is a more extensive application of the Gaussian wave-packet states to describe a variety of quantum dynamical processes of the atomic and molecular systems. Though quantum mechanically it is not limited to use the wave-packet states to describe the quantum control process of the reversible and unitary halting protocol and to investigate the mechanism of the state-locking pulse field [1], the quantum mechanical wave-packet states not only provide an intuitive picture for understanding the mechanisms of the reversible and unitary state-insensitive halting protocol and the state-locking pulse field but also simplify the quantitative and exact calculation for the quantum control process in the physical system of the halting-qubit atom. According to the reversible and unitary state-insensitive halting protocol [1] the wave-packet motional state of the halting-qubit atom should have a small spread, the wave-packet amplitude of the motional state decays quickly with the deviation from the wave-packet center and it is close to zero outside the effective spread. Obviously, such a wave-packet picture is close to a particle picture, which is just required by the reversible and unitary state-insensitive halting protocol. A Gaussian wave-packet state satisfies this requirement very much. It can have a very narrow linewidth or wave-packet spread and its amplitude at a position deviating from the center-of-mass position decays rapidly and exponentially with square of distance between the position and the center-of-mass position. A Gaussian wave-packet state is described completely by the three basic parameters: the center-of-mass position, the mean momentum, and the complex linewidth [2, 5, 6, 7, 8]. Then a Gaussian wave-packet state is simple and easy to be treated in theory and it is also easily manipulated and controlled in experiment. On the other hand, it is well known that the ground state of a harmonic oscillator is a Gaussian wave-packet state [2], while at the initial time of the quantum control process the halting-qubit atom is prepared to be in the ground state of harmonic oscillator [1]. Thus, it is convenient and natural to choose the Gaussian wave-packet state to describe and calculate quantitatively the quantum control process. Due to the fact that a Gaussian wave-packet state has these advantages it should be better to keep Gaussian shape of the motional state of the halting-qubit atom unchanged in the whole quantum control process.

Unitary manipulation in time and space in a quantum system plays a key role in implementing the reversible and unitary state-insensitive halting protocol and realizing the efficient quantum search process [1, 12]. It is known that the quantum control process to realize the reversible and unitary state-insensitive halting protocol is a unitary evolution process in time and space [1]. One of the key components of the quantum control process is the time- and space-compressing processes which are realized by the unitary decelerating and accelerating processes of the halting-qubit atom. It has been shown [8] that the unitary decelerating and accelerating processes for a free particle moving

in space can be realized by the stimulated Raman adiabatic passage (STIRAP) method [9]. In the ideal or near ideal adiabatic condition the STIRAP pulse sequence can transfer completely one Gaussian wave-packet motional state of the free atom to another [8]. It is known that the unitary propagator of a quadratic Hamiltonian can generally keep Gaussian shape unchanged for a Gaussian wave-packet state when the state is acted on by the unitary propagator [5, 6, 13, 14]. The Hamiltonian of the atom in the presence of the STIRAP pulse sequence is completely different from a conventional quadratic Hamiltonian. But it is surprising that the unitary propagator corresponding to the Hamiltonian of the atom in the presence of the STIRAP pulse sequence does not yet change Gaussian shape of an atomic Gaussian wave-packet motional state when the atom is irradiated by the STIRAP pulse sequence in the ideal or near ideal adiabatic condition. The STIRAP-based unitary decelerating and accelerating processes can manipulate the center-of-mass position and momentum of a Gaussian wave-packet state of a free atom in time and space, but it generally does not control the complex linewidth of the Gaussian wave-packet state because the imaginary part of the complex linewidth always increases linearly with the time period of the decelerating or accelerating process. One advantage to use the laser light fields such as the STIRAP pulse sequence to manipulate an atomic Gaussian wave-packet motional state is that the space-selective manipulation of the atom can be implemented easily. On the other hand, a general quadratic Hamiltonian can be used to manipulate not only the complex linewidth but also the center-of-mass position and momentum of a Gaussian wave-packet state. This will be investigated in detail in the paper.

This paper is devoted to the construction of the state-selective trigger pulse and the unitary manipulation of a Gaussian wave-packet state in time and space by using a general quadratic Hamiltonian. A state-selective trigger pulse is a key component to realize both the reversible and unitary state-insensitive halting protocol and the efficient quantum search process based on the unitary quantum dynamics in time and space [1, 12]. It is generally related to unitary manipulation and control in time and space of the atomic center-of-mass and internal motions as well as the coupling of the two motions. In the quantum control process the state-selective trigger pulse is used to transfer the ground motional state of the halting-qubit atom in the harmonic potential field to a standard coherent state [10, 11] with a higher motional energy. It is known that a standard coherent state of harmonic oscillator is a Gaussian wave-packet state [13]. There is a requirement that the coherent-state excitation process induced by the state-selective trigger pulse be dependent on the atomic internal state and keep Gaussian shape of the atomic motional state unchanged. One of the convenient methods to construct the state-selective trigger pulse could be that the Hamiltonian of the halting-qubit atom in the presence of state-selective trigger pulse is prepared to be a state-dependent quadratic Hamiltonian. A general quadratic Hamiltonian of an atomic system may be generated either by an external static electric or magnetic field or by the externally applied electromagnetic wave field. An external static electric or magnetic field usually does not induce the transitions of the atomic internal electronic or nuclear spin states. There-

fore, these uniform static electric and magnetic fields are not suited to build up the internal-state-selective trigger pulse. Generally, the construction of the state-selective trigger pulse needs to use the external electromagnetic wave field.

2. The constructions for the state-selective trigger pulse

The state-selective trigger pulse is one of the key components to construct the quantum control process to simulate the reversible and unitary state-insensitive halting protocol. It is used to control the halting operation in the reversible and unitary state-insensitive halting protocol. The properties for the state-selective trigger pulse have been described in detail in the paper [1]. There are two basic properties for the state-selective trigger pulse. One of which is that the trigger pulse is internal-state-selective. Only when the halting-qubit atom is in the given internal state can the state-selective trigger pulse take a real action on the atom and excite the ground motional state of the atom to the coherent state. Another is that the state-selective trigger pulse can keep Gaussian shape unchanged for the atomic motional state. Under the state-selective trigger pulse the initial motional state of the atom could be converted into a standard coherent state of harmonic oscillator with a higher motional energy. In the quantum control process the initial motional state generally is prepared to be the ground state of the harmonic oscillator — the halting-qubit atom in the left-hand harmonic potential well [1]. It is well known that the ground state of a harmonic oscillator is a Gaussian wave-packet state [2] and so is a standard coherent state of the harmonic oscillator [13]. Therefore, it is clear that the state-selective trigger pulse should generate a unitary propagator for the halting-qubit atom such that the unitary propagator does not change the Gaussian wave-packet shape of the motional state of the halting-qubit atom. According to these properties one may construct explicitly such a state-selective trigger pulse for the reversible and unitary halting protocol in the physical system of the halting-qubit atom in the harmonic potential well. One could choose a sequence of laser light pulses as the state-selective trigger pulse in the atomic physical system. But there could also be other schemes to construct the state-selective trigger pulse.

2.1. The non-internal-state-selective excitation process

Before the internal-state-selective excitation process of an atom in a harmonic potential field is discussed in next subsections, this subsection is devoted to the investigation for the non-internal-state-selective excitation process of the atom. If there is not the internal-state-selective requirement, it is usually easy to excite a coherent state from the ground state of a harmonic oscillator. In principle one of the simplest methods to generate non-selectively a coherent state from the ground state of a harmonic oscillator is to apply directly an external classical electric or magnetic field to the harmonic oscillator, so that an extra force generated by the external electric or magnetic field is exerted on the harmonic oscillator. Then it can prove [14, 15, 16, 17] that this extra force can drive the atom from the ground state to the coherent state with a higher motional energy. For example, for a trapped atomic ion in the harmonic poten-

tial well one may apply a spatially uniform driving electric field to the trapped atomic ion. The experiments to confirm the simple method have been carried out in the trapped ion systems [18, 19, 20]. Since the classical driving field does not excite any internal state of the trapped atom in the harmonic potential well and the trapped atom in any internal state can be excited from the ground motional state to the coherent state by the external driving field, this method is an internal-state-independent excitation method to generate the coherent state. The coherent-state excitation process may be described simply by the Hamiltonian of the atom in the harmonic potential well and in the presence of the external driving field,

$$H(t) = H_0 + H(r) + H_1(t), \tag{1}$$

where the Hamiltonian H_0 describes the center-of-mass motion of the atom in the harmonic potential well and in the absence of the external driving field,

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, \tag{2}$$

here the oscillatory frequency $\omega = \omega(t)$ of the harmonic oscillator may be time-dependent or may not, the term $H(r)$ is the internal Hamiltonian of the atom which describes the internal electronic (or nuclear spin) motion of the atom, and $H_1(t)$ is the interaction between the atom in the harmonic potential well and the external driving field. One may not consider the internal Hamiltonian $H(r)$ of the atom in the internal-state-independent excitation process, since the excitation process is not dependent on any internal state of the atom. One of the simplest interactions is $H_1(t) = f(t)x$ [14, 15, 16, 17]. It indicates that there is the force exerted on the harmonic oscillator to generate the coherent state. This linear interaction $H_1(t)$ may be generated by simply applying the driving electric or magnetic field to the harmonic potential well in a similar way to generating the harmonic potential $m\omega^2 x^2/2$. For example, if the atom is a charged atomic ion with charge q, the interaction may be written as $H_1(t) = -qxE(t)$ for the spatially uniform and time-dependent driving electric field $E(t)$ and hence the forced function $f(t)$ is given by $f(t) = -qE(t)$. The atom is usually in some hyperfine ground electronic (or nuclear spin) state before the external driving electric field is applied. However, when the atom is irradiated by the suitable external electromagnetic field, it may jump to an atomic excited state or another hyperfine ground state. These internal electronic (or nuclear spin) state transitions of the atom have their own transition frequencies. Suppose that the minimum transition frequency among these internal-state transition frequencies is much greater than the oscillatory frequency of the external driving electric field $E(t)$. Then it is impossible for the external driving electric field to induce the atom to make a transition between the atomic internal states. During the external driving field the atom stays in its initial internal state. Thus, the coherent-state excitation process of the atom under the external driving electric field is independent of the atomic internal states and hence it is internal-state-independent. On the other hand, while the coherent-state excitation process is

independent of any atomic internal state, the driving electric field $E(t)$ must be able to induce effectively the transitions between the vibrational energy levels of the atom in the harmonic potential well. The vibrational energy levels generally have a much smaller energy-level space than the atomic internal energy levels. If now the oscillatory frequency of the external electric field is set to be near the oscillatory frequency of the harmonic oscillator — the halting-qubit atom in the harmonic potential well, then the external driving electric field may induce the atom to make a transition between the vibrational energy levels, while the atomic internal states are not affected by the driving electric field. Since any internal state of the atom remains unchanged during the coherent-state excitation process by the external driving electric field, the atomic internal Hamiltonian $H(r)$ may be omitted from the total Hamiltonian $H(t)$ of Eq. (1). Then in this case the Hamiltonian to describe the time evolution process of the atom in the harmonic potential well and in the presence of the external driving electric field is reduced to the form

$$H(t) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + f(t)x. \tag{3}$$

This is the Hamiltonian of a time-dependent forced harmonic oscillator [13, 14, 15, 16, 17, 21]. If now the initial motional state of the atom is the ground state of the harmonic oscillator, then the unitary propagator of the Hamiltonian $H(t)$ (3) will convert the ground state into a coherent state [17], indicating that the external driving electric field can transfer the ground state to a coherent state. Though the construction for the state-selective trigger pulse does not use the state-independent excitation method as mentioned above, this method may be useful when one manipulates unitarily a Gaussian wave-packet state in time and space by a unitary propagator of a quadratic Hamiltonian, as can be seen in section 4. This is because the Hamiltonian (3) is really the specific form of a general quadratic Hamiltonian. The internal-state-independent excitation method is also useful in the quantum control process [1]. When the halting-qubit atom returns back to the left-hand harmonic potential well in the quantum control process, its motional state needs to be transferred back to the original ground motional state. Then the state-independent excitation method may help the quantum control process to realize such a state transfer.

For a neutral atom system both the harmonic potential field and the external force field may be generated by applying an external driving electric or magnetic field. It is known that an atom can have an induced electric dipole moment in the presence of an external electric field [2]. Then a potential energy $V(x,t)$ of the atom can be generated due to the induced electric dipole moment in the external electric field $E(x,t)$, which may be expressed as $V(x,t) = \frac{1}{2}\alpha|E(x,t)|^2$, where α is the atomic polarizability [2]. And hence the interaction $H_1(t)$ in Eq. (1) could be obtained from the potential energy $V(x,t)$. Obviously, this atomic potential energy is internal-state-independent. Thus, the time-dependent Hamiltonian $H(t)$ of Eq. (3) for the neutral atom in a harmonic potential well may also be generated by applying a suitable time- and space-dependent electric field $E(x,t)$ to the atom. For example, if the external electric field $E(x,t) \propto ax + b$,

then the potential energy $V(x,t)$ contains the linear interaction $H_1(t) \propto x$. On the other hand, a nuclear spin or a neutral atom which has an intrinsic magnetic dipole moment may generate a state-dependent force in an inhomogeneous magnetic field [50]. A nuclear spin or atom in an external magnetic field can generate the Zeeman effect [2, 50] and the spin energy level is given by $E_m = -\gamma B(x,t)\hbar m$, where $B(x,t)$ is the external magnetic field strength, m the spin magnetic quantum number, and γ the gyromagnetic ratio of the spin. Thus, the interaction between the spin and the external magnetic field is given by $H_1(t) = -\gamma\hbar B(x,t)m$. When the external magnetic field is not uniform, for example, $B(x,t) \propto x$, the spin will be acted on by an external force generated by the magnetic field $B(x,t)$ if the spin magnetic quantum number $m \neq 0$. Since the quantum number m marks the spin quantum state, the spin state with $m = 0$ is not acted on by the external magnetic field, while all those spin states with $m \neq 0$ undergo the external magnetic field. Therefore, the spin interaction $H_1(t)$ may be time- and state-dependent. Then an inhomogeneous external magnetic field could act as a state-selective trigger pulse if the halting-qubit atom is chosen as a nuclear spin (its spin quantum number is even) or a neutral atom with an intrinsical magnetic dipole moment and the external magnetic field is designed suitably.

2.2. The off-resonance selective excitation method

A general method to construct the state-selective trigger pulse is involved in manipulating and controlling in time and space the center-of-mass motion and the internal electronic (or spin) motion of an atom as well as the coupling between the center-of-mass and the internal motion. Since the coherent-state selective excitation process is dependent upon the specific internal state of the atom during the state-selective trigger pulse, the atomic center-of-mass motion must be coupled with the internal electronic (or spin) motion of the atom in the excitation process. Coherently manipulating the center-of-mass and the internal motion of the atom as well as the interaction between the two motions is the fundament for realizing the atomic laser light cooling and trapping in an atomic ensemble [22], implementing the quantum computation [23], and preparing and transferring various quantum states in an atom-ionic system in a harmonic potential field [24, 25, 26, 27]. The state-selective trigger pulse is also closely related to the coupling between the atomic center-of-mass and internal motions. Laser light is a general technique to realize the coupling between the internal and the center-of-mass motion of an atom. In particular, as one of the most useful double-photon excitation methods the stimulated Raman adiabatic passage (STIRAP) method [9] has been extensively used to coherently manipulate the center-of-mass and the internal motion of an atom and also used to create and control the coupling between the atomic center-of-mass and internal motions in an atomic ensemble in quantum interference experiments [28, 29]. The largest advantage of the STIRAP method is that the STIRAP method can achieve a complete state-transfer efficiency in theory and is tolerant to the experimental imperfection and can avoid the spontaneous emission generated by

atomic excited states. For example, a high internal-state transfer efficiency is achieved experimentally by the STIRAP method in trapped ions [30]. On the other hand, the conventional Raman double-photon laser light techniques also have been used to prepare the ground motional state and various quantum coherent states and to transfer one atomic internal state to another in an atomic ion system in a harmonic potential field [18, 27, 31, 32, 33]. The STIRAP method usually uses a pair of Raman adiabatic laser light beams to couple the atomic center-of-mass motional state and internal electronic states (or spin polarization states) so as to realize the interaction between the atomic center-of-mass and internal motions. In order to generate effectively and internal-state-selectively the coupling between the two atomic motions the laser light electromagnetic field should have an oscillatory frequency close to the resonance frequency of a given pair of the internal electronic (or spin) states of the atom and far from those resonance frequencies of any atomic internal-state transitions other than this given resonance frequency. With this frequency setting the electromagnetic field can induce only the transition between the given pair of the atomic internal states and does not induce any other atomic internal-state transitions when the amplitude of the electromagnetic field is not very large. Thus, this is an internal-state-selective excitation process. This state-selective excitation process may be described by the unitary quantum dynamics in theory. Since this is an internal-state-selective excitation process, one must consider the atom to be a multi-level physical system. The simplest case is that the atom is a three-level physical system and two of the three internal energy levels are irradiated selectively by the external electromagnetic field. Then in this case the atomic internal Hamiltonian $H(r)$ may be simply written as

$$H(r) = E_0|\psi_0(r)\rangle\langle\psi_0(r)| + E_1|\psi_1(r)\rangle\langle\psi_1(r)| + E_2|\psi_2(r)\rangle\langle\psi_2(r)| \qquad (4)$$

where E_k and $|\psi_k(r)\rangle$ $(k = 0, 1, 2)$ are the $k-$th eigenvalue and eigenstate of the internal Hamiltonian $H(r)$, respectively, that is, $H(r)|\psi_k(r)\rangle = E_k|\psi_k(r)\rangle$. For example, the eigenstates $|\psi_0(r)\rangle$ and $|\psi_1(r)\rangle$ may be taken as the two hyperfine ground electronic states $|g_0\rangle$ and $|g_1\rangle$ of an atom, respectively, while $|\psi_2(r)\rangle$ is taken as the atomic excited state $|e\rangle$. Here suppose that the external electromagnetic field is applied selectively to the two atomic internal energy levels $|g_0\rangle$ and $|e\rangle$, while the internal state $|g_1\rangle$ is not affected by the electromagnetic field. On the other hand, the semiclassical theory of the electromagnetic radiation and the electric dipole approximation are still suited to describe the state-selective excitation process in the physical system of the atom plus the electromagnetic field [2, 34]. In the electric dipole approximation the interaction between the atom and an externally applied electromagnetic field can be expressed as $H_1(t) = -D \cdot \mathbf{E}(x, t)$, where D is the electric dipole operator of the atom, $E(x, t)$ the time-dependent electric field of the externally applied electromagnetic field, and the coordinate x the center-of-mass position of the atom. The selective excitation method may use either the single- or double-frequency (or double-photon) or even multi-photon excitation method. For the single- and double-frequency selective excitation processes the external electromagnetic

fields are respectively written as

$$E(x,t) = \frac{1}{2}E_{L0}(t)\exp[i(k_{L0}.x - \omega_{L0}t)] + C.C. \tag{5}$$

and

$$E(x,t) = \frac{1}{2}E_{L0}(t)\exp[i(k_{L0}.x - \omega_{L0}t)]$$

$$+\frac{1}{2}E_{L1}(t)\exp[i(k_{L1}.x - \omega_{L1}t)] + C.C., \tag{6}$$

where $E_{Lk}(t)$, k_{Lk}, and ω_{Lk} $(k = 0, 1)$ are the complex amplitude, wavevector, and oscillatory frequency of the laser light beam, respectively, and $C.C.$ stands for the complex (or Hermite) conjugate term. In the single-frequency selective excitation process only one laser light beam $(E_{L0}(t), k_{L0}, \omega_{L0})$ is selectively applied to the two internal energy levels $|g_0\rangle$ and $|e\rangle$, while in the double-frequency selective excitation process a pair of the laser light beams $(E_{L0}(t), k_{L0}, \omega_{L0})$ and $(E_{L1}(t), k_{L1}, \omega_{L1})$ are selectively applied to the two internal states $|g_0\rangle$ and $|e\rangle$ simultaneously. Note that the atomic internal state $|g_1\rangle$ is not affected by any laser light beam in the single- and double-frequency excitation processes. There is a slight difference between the double-frequency selective excitation method here and the conventional STIRAP experiment. In the conventional STIRAP experiment a pair of the Raman laser light beams usually are applied to two different transitions linking the two different ground internal states $|g_0\rangle$ and $|g_1\rangle$ to the same excited state $|e\rangle$, respectively. Now the electric dipole interaction $H_1(t)$ between the atom and the electric field $E(x,t)$ of the external electromagnetic field is given explicitly by [24, 34, 35]

$$H_1(t) = \hbar(\Omega_{L0}(t)I^+ + \Omega^*_{L0}(t)I^-)\cos(k_{L0}x - \omega_{L0}t)$$

$$= \hbar\Omega_0(t)\{I^+\exp[i(k_{L0}x - \omega_{L0}t - \varphi_0(t))]$$

$$+I^-\exp[-i(k_{L0}x - \omega_{L0}t - \varphi_0(t))]\} \tag{7}$$

for the single-frequency selective excitation process, where the second equality is obtained in the rotating wave approximation, and in the rotating wave approximation for the double-frequency selective excitation process,

$$H_1(t) = \hbar\Omega_0(t)\{I^+\exp[i(k_0x - \omega_0t - \varphi_0(t))]$$

$$+I^-\exp[-i(k_0x - \omega_0t - \varphi_0(t))]\}$$

$$+\hbar\Omega_1(t)\{I^+\exp[i(k_1x - \omega_1t - \varphi_1(t))]$$

$$+I^-\exp[-i(k_1x - \omega_1t - \varphi_1(t))]\} \tag{8}$$

where the two laser light beams may be either counterpropagating (k_0 and k_1 have the opposite signs) or copropagating (k_0 and k_1 have the same signs), the

amplitude $\Omega_{Ll}(t) = \Omega_l(t)\exp[-i\varphi_l(t)]$ $(l = 0, 1)$, and the atomic internal-state operators are defined by

$$2I_z = (I^1 - I^0),\ I^0 = |g_0\rangle\langle g_0|,\ I^1 = |e\rangle\langle e|,\ I^+ = |e\rangle\langle g_0|,\ I^- = |g_0\rangle\langle e|.$$

These interactions (7) and (8) are similar to those in the Jaynes–Cummings model [35] of an atom plus electromagnetic field system. Now the total Hamiltonian (1) can be given explicitly if the internal Hamiltonian $H(r)$ (4) and the electric dipole interaction $H_1(t)$ of Eq. (7) or (8) are inserted into Eq. (1). It should be pointed out that it is possible to apply an extra laser light beam for each laser light beam above to compensate the rotating-wave approximation. If each laser light beam in the selective excitation process above is replaced with a pair of the laser light beams with the orthogonal electric field vectors and the suitable phases, one can eliminate the rotating-wave approximation. This means that the Hamiltonian (8) may be constructed exactly.

It is known that the quantum behavior of an atom in the harmonic potential field and in the presence of the external electromagnetic field may be described by the complete set of the product basis states $\{|\Psi_{nk}(x, r)\rangle\}$,

$$|\Psi_{nk}(x, r)\rangle = |\psi_n(x)\rangle|\psi_k(r)\rangle, \tag{9}$$

where $|\psi_n(x)\rangle$ is an eigenstate of the Hamiltonian H_0 (2) of the harmonic oscillator, which is used to describe the atomic center-of-mass motion, while $|\psi_k(r)\rangle$ is an eigenstate of the internal Hamiltonian $H(r)$, as can be seen in Eq. (4), which is used to describe the internal electronic (or spin) motion of the atom. Since the external electromagnetic field is applied to only the two internal states $|g_0\rangle$ and $|e\rangle$, the time evolution process of the atom in the harmonic potential well and in the presence of the external electromagnetic field is described by the unitary propagator

$$U(t) = T\exp\{-\frac{i}{\hbar}\int_0^t dt' H(t')\} = \exp\{-i\frac{E_1 t}{\hbar}|g_1\rangle\langle g_1|\}U_L(t) \tag{10}$$

where the total Hamiltonian $H(t)$ is given by Eq. (1) and the unitary propagator $U_L(t)$ is defined as

$$U_L(t) = T\exp\{-\frac{i}{\hbar}\int_0^t dt' H_L(t')\} \tag{10a}$$

with the Hamiltonian $H_L(t)$ given by

$$H_L(t) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + E_0 I^0 + E_2 I^1 + H_1(t). \tag{11}$$

Though the Hamiltonian $H_L(t)$ is involved in only the two-state subspace span by the internal states $|g_0\rangle$ and $|e\rangle$, it contains the term H_0 (2) and consequently the unitary propagator $U_L(t)$ still can affect the atomic product state $|\Psi(x, t)\rangle|g_1\rangle$, where $|\Psi(x, t)\rangle$ is an atomic center-of-mass motional state. For example, the unitary propagator $U_L(t)$ may have a significant effect on the product

state $|\Psi(x,t)\rangle|g_1\rangle$ if the motional state $|\Psi(x,t)\rangle$ is a superposition state. Actually, the Hamiltonian $H_L(t)$ could not be considered as the Hamiltonian of a two-level system consisting of the internal states $|g_0\rangle$ and $|e\rangle$. However, the unitary propagator $U_L(t)$ can generate only a global phase factor for the product state $|\Psi(x,t)\rangle|g_1\rangle$ if the motional state $|\Psi(x,t)\rangle$ is an energy eigenstate $|\psi_n(x)\rangle$ of the harmonic oscillator. Now suppose that at the initial time the halting-qubit atom is in the product state $|\psi_0(x)\rangle|g_1\rangle$, where $|\psi_0(x)\rangle$ is the ground motional state of the harmonic oscillator. Obviously, the atom is still in the product state $|\psi_0(x)\rangle|g_1\rangle$ if it is acted on by the unitary propagator $U(t)$ of Eq. (10). However, after the atom is transferred to another internal state $|g_0\rangle$ or $|e\rangle$ from the initial internal state $|g_1\rangle$, it could be excited to a motional state that has a higher motional energy than the ground motional state by the unitary propagator $U(t)$. This means in the case that the unitary propagator $U(t)$ is really state-selective, and the excitation process may be described directly by the Hamiltonian $H_L(t)$ of Eq. (11). Now one wants to design the external electromagnetic field $E(x,t)$ of Eq. (5) or (6) such that the atomic ground motional state is transferred to a standard coherent state of harmonic oscillator by the unitary propagator $U(t)$ of Eq. (10) or $U_L(t)$ of Eq. (10a). The conventional Raman laser light beams [18, 19, 31, 32, 33] are often used to generate selectively the coherent state. They may be understood intuitively below. At the starting time of the excitation process the atom is in the ground motional state $|\psi_0(x)\rangle$ and the internal state $|g_0\rangle$. Then the first Raman laser light beam excites the atom from the internal state $|g_0\rangle$ to the excited state $|e\rangle$, while the second Raman laser light beam induces the atom from the excited state $|e\rangle$ to jump back to the original internal state $|g_0\rangle$, and at the same time the initial ground motional state $|\psi_0(x)\rangle$ is changed to a coherent state during the excitation process. Although the atomic internal state $|g_0\rangle$ is not changed after the excitation process, the atomic motional state is changed from the initial ground state $|\psi_0(x)\rangle$ to the coherent state. Obviously, this coherent state is indirectly generated through the internal-state transfer pathway $|g_0\rangle \rightarrow |e\rangle \rightarrow |g_0\rangle$ by the two Raman laser light beams. Therefore, this is an internal-state-dependent excitation process.

It could be convenient to investigate the internal-state-selective excitation process based on the Raman laser light beams in the Heisenberg picture which is often used in the laser spectroscopy [34]. The excitation process is involved in only the two-state subspace span by the internal states $|g_0\rangle$ and $|e\rangle$ and is governed by the Hamiltonian $H_L(t)$ of Eq. (11). Then all those operators appearing in the Hamiltonian $H_L(t)$ of Eq. (11) are defined as $A(t) = U_L(t)^+ A U_L(t)$ in the Heisenberg picture, where the operator A may be I^0, I^1, I^+, I^-, x, and p. The dynamical equations for these operators in the Heisenberg picture are given by

$$i\hbar\frac{d}{dt}A(t) = [A(t), H_L(t)]. \tag{12}$$

Below consider a general double-frequency excitation method including those using the conventional Raman laser light beams. By inserting the Hamiltonian $H_L(t)$ of Eq. (11) with the electric dipole interaction $H_1(t)$ of Eq. (8) into the

Heisenberg equations (12) one obtains [34]

$$i\hbar\frac{d}{dt}I^+(t) = -\hbar\omega_a I^+(t) + 2\hbar\Omega_0(t)I_z(t)\exp[-ik_0x(t)]\exp[i(\omega_0 t+\varphi_0)]$$

$$+2\hbar\Omega_1(t)I_z(t)\exp[-ik_1x(t)]\exp[i(\omega_1 t+\varphi_1)], \tag{12a}$$

$$i\hbar\frac{d}{dt}I^-(t) = \hbar\omega_a I^-(t) - 2\hbar\Omega_0(t)I_z(t)\exp[ik_0x(t)]\exp[-i(\omega_0 t+\varphi_0)]$$

$$-2\hbar\Omega_1(t)I_z(t)\exp[ik_1x(t)]\exp[-i(\omega_1 t+\varphi_1)], \tag{12b}$$

$$\begin{aligned} i\hbar\frac{d}{dt}I_z(t) = {} & \hbar\Omega_0(t)\{I^+(t)\exp[i(k_0x(t)-\omega_0 t-\varphi_0)] \\ & -I^-(t)\exp[-i(k_0x(t)-\omega_0 t-\varphi_0)]\} \\ & +\hbar\Omega_1(t)\{I^+(t)\exp[i(k_1x(t)-\omega_1 t-\varphi_1)] \\ & -I^-(t)\exp[-i(k_1x(t)-\omega_1 t-\varphi_1)]\}, \end{aligned} \tag{12c}$$

and

$$\begin{aligned} \frac{d}{dt}p(t) = {} & -m\omega^2x(t) \\ & -i\hbar k_0\Omega_0(t)\{I^+(t)\exp[i(k_0x(t)-\omega_0 t-\varphi_0)] \\ & -I^-(t)\exp[-i(k_0x(t)-\omega_0 t-\varphi_0)]\} \\ & -i\hbar k_1\Omega_1(t)\{I^+(t)\exp[i(k_1x(t)-\omega_1 t-\varphi_1)] \\ & -I^-(t)\exp[-i(k_1x(t)-\omega_1 t-\varphi_1)]\}, \end{aligned} \tag{12d}$$

$$\frac{d}{dt}x(t) = p(t)/m, \tag{12e}$$

where $E_0I^0 + E_2I^1 = \alpha_0 E + \hbar\omega_a I_z$, and E is the 2×2 unit operator, $\alpha_0 = (E_2+E_0)/2$, $\hbar\omega_a = (E_2-E_0)$, and ω_a is the resonance frequency of the two atomic internal energy levels $|g_0\rangle$ and $|e\rangle$. The former three equations mainly describe the atomic internal motion and the coupling between the internal and center-of-mass motions of the atom, while the rest two equations mainly describe the center-of-mass motion and the coupling of the two motions. The Heisenberg equation set of Eqs. (12) describes completely the time evolution process of the atom, which is involved in the atomic internal and center-of-mass motions as well as the coupling of the two motions. In the absence of the external electromagnetic field the Heisenberg equations (12) have a simple solution [34]:

$$I^{\pm}(t) = I^{\pm}(0)\exp[\pm i\omega_a t], \quad I_z(t) = I_z(0),$$

$$a(t)^+ = a(0)^+\exp(i\omega t), \quad a(t) = a(0)\exp(-i\omega t),$$

where the creation (a^+) and annihilation (a) operators defined through

$$x(t) = \sqrt{\frac{\hbar}{2m\omega}}(a(t)^+ + a(t)),\ p(t) = i\sqrt{\frac{1}{2}\hbar\omega m}(a(t)^+ - a(t)). \tag{13}$$

This simple solution is called the uncoupling solution to the Heisenberg equations (12).

Now the off-resonance excitation method is introduced below. The off-resonance excitation means that the two atomic internal states $|g_0\rangle$ and $|e\rangle$ are irradiated by weak and off-resonance laser light beams. This also means that for the off-resonance excitation using the Raman adiabatic laser light beams the Rabi frequencies $\Omega_0(t)$ and $\Omega_1(t)$ of the two Raman adiabatic laser light beams are slowly varying and much less than the frequency offsets $(\omega_a - \omega_0)$ and $(\omega_a - \omega_1)$. Obviously, in the off-resonance excitation the solution to the Heisenberg equations (12) should be close to the uncoupling solution and hence the operators $\{I^{\pm}(t)\exp[\mp i\omega_a t]\}$ should be close to the operator $I^{\pm}(0)$. Therefore, the time derivatives of the operators $\{I^{\pm}(t)\exp[\mp i\omega_a t]\}$ are close to zero. Now one may make a unitary transformation [33]:

$$\hat{I}^{\pm}(t) = \exp(-i\omega_a I_z(t)t)I^{\pm}(t)\exp(i\omega_a I_z(t)t) = I^{\pm}(t)\exp[\mp i\omega_a t]. \tag{14}$$

Then the first two Heisenberg equations (12a) and (12b) are reduced respectively to the forms

$$\frac{d}{dt}\hat{I}^{+}(t) = -2i\Omega_0(t)I_z(t)\exp[-ik_0x(t)]\exp[-i(\omega_a - \omega_0)t + i\varphi_0]$$

$$-2i\Omega_1(t)I_z(t)\exp[-ik_1x(t)]\exp[-i(\omega_a - \omega_1)t + i\varphi_1)] \tag{15a}$$

and

$$\frac{d}{dt}\hat{I}^{-}(t) = 2i\Omega_0(t)I_z(t)\exp[ik_0x(t)]\exp[i(\omega_a - \omega_0)t - i\varphi_0]$$

$$+2i\Omega_1(t)I_z(t)\exp[ik_1x(t)]\exp[i(\omega_a - \omega_1)t - i\varphi_1]. \tag{15b}$$

Obviously, the time derivatives of the operators $\{\hat{I}^{\pm}(t)\}$ are close to zero when the Rabi frequencies $\Omega_0(t)$ and $\Omega_1(t)$ are close to zero. If the frequency offsets $\{|(\omega_a - \omega_l)|\}$ are much greater than the Rabi frequencies $\{\Omega_l(t)\}$ $(l = 0, 1)$, the atomic motional velocity $p(t)/m$ is small in the harmonic potential well, and the time varying of the Rabi frequencies is slow (the Raman adiabatic laser light beams satisfy the condition), then the operators $\{\hat{I}^{\pm}(t)\}$ may be obtained approximately by integrating the two equations (15), respectively, because the oscillatory terms $\exp[\pm i(\omega_l - \omega_a)t]$ $(l = 0, 1)$ will generate a dominating contribution to the two integrals. Now by integrating first the two equations (15), then making integration by part, and then by using Eq. (14) one obtains the two operators:

$$I^{+}(t) - \exp(i\omega_a t)I^{+}(0) = \frac{2\exp(i\varphi_0)}{(\omega_a - \omega_0)}\Omega_0(t)I_z(t)\exp[-ik_0x(t)]\exp[i\omega_0 t]$$

$$+\frac{2\exp(i\varphi_1)}{(\omega_a-\omega_1)}\Omega_1(t)I_z(t)\exp[-ik_1x(t)]\exp[i\omega_1 t]+E(I^+), \tag{16a}$$

$$I^-(t)-\exp(-i\omega_a t)I^-(0)=\frac{2\exp(-i\varphi_0)}{(\omega_a-\omega_0)}\Omega_0(t)I_z(t)\exp[ik_0x(t)]\exp[-i\omega_0 t]$$

$$+\frac{2\exp(-i\varphi_1)}{(\omega_a-\omega_1)}\Omega_1(t)I_z(t)\exp[ik_1x(t)]\exp[-i\omega_1 t]+E(I^-) \tag{16b}$$

For simplicity, here suppose that the phases φ_0 and φ_1 are time-independent and at the initial time both the amplitudes of the two Raman adiabatic laser light beams are zero, that is, $\Omega_0(0)=\Omega_1(0)=0$. Since these terms $\{\exp(\pm i\omega_a t)I^{\pm}(0)\}$ are of the uncoupling solution, the right-hand sides of the equations (16) measure the deviation of the real solution from the uncoupling solution. The terms $E(I^+)$ and $E(I^-)$ in Eq. (16) are error terms which are higher-order terms of the frequency offsets ($(\omega_a-\omega_l)^{-2}$ $(l=0,1)$, $(\omega_a-\omega_l)^{-1}(\omega_a-\omega_l)^{-1}$, etc.). The error terms $E(I^+)$ and $E(I^-)$ can be neglected in the first-order approximation. That is, only the first-order terms which are proportional to the inverse frequency offsets ($|\omega_a-\omega_l|^{-1}$, $l=0,1$) are retained in the operators $\{I^{\pm}(t)\}$ in the first-order approximation. Now inserting the operators $\{I^{\pm}(t)\}$ (16a) and (16b) of the first-order approximation into the Heisenberg equation (12c) one obtains, up to the first-order approximation,

$$I_z(t)-I_z(0)=-\frac{\Omega_0(t)\exp[-i\varphi_0]}{(\omega_a-\omega_0)}I^+(0)\exp[ik_0x(t)]\exp[i(\omega_a-\omega_0)t]$$

$$-\frac{\Omega_1(t)\exp[-i\varphi_1]}{(\omega_a-\omega_1)}I^+(0)\exp[ik_1x(t)]\exp[i(\omega_a-\omega_1)t]+C.C. \tag{16c}$$

where $C.C.$ stands for the hermite conjugate of the first two terms on the right-hand side of Eq. (16c). The two Heisenberg equations (12d) and (12e) then can be solved with the help of the first-order approximation operators $\{I^{\pm}(t)\}$ (16a) and (16b) and $I_z(t)$ (16c) and one obtains, in the first-order approximation,

$$a(t)^+-\exp(i\omega t)a(0)^+=iI_z(0)\sqrt{\frac{2\hbar}{m\omega}}\int_0^t dt'\{\Omega_{eff}(t')$$

$$\times\exp[-i\omega(t'-t)]\sin[\Delta\omega t'+\Delta\varphi]\}, \tag{16d}$$

$$a(t)-\exp(-i\omega t)a(0)=-iI_z(0)\sqrt{\frac{2\hbar}{m\omega}}\int_0^t dt'\{\Omega_{eff}(t')$$

$$\times\exp[i\omega(t'-t)]\sin[\Delta\omega t'+\Delta\varphi]\}, \tag{16e}$$

where the Lamb-Dicke limit $||\Delta kx(t)|| << 1$ and the condition $m\omega^2/\hbar >> |\Delta k\Omega_{eff}(t)|$ have been used; $\Delta k=k_0-k_1$, $\Delta\omega=\omega_0-\omega_1$, $\Delta\varphi=\varphi_0-\varphi_1$; and the parameter $\Omega_{eff}(t)$ is given by

$$\Omega_{eff}(t)=\frac{2k_0\Omega_0(t)\Omega_1(t)}{(\omega_a-\omega_1)}-\frac{2k_1\Omega_0(t)\Omega_1(t)}{(\omega_a-\omega_0)}.$$

The condition $m\omega^2 >> |\hbar\Delta k\Omega_{eff}(t)|$ means that the effect of the Raman laser light beams on the harmonic potential field $(m\omega^2 x^2/2)$ can be neglected. Now by inserting the first-order approximation operators $\{I^{\pm}(t)\}$ of Eq. (16a) and (16b) into the electric dipole interaction $H_1(t)$ (8) and then inserting Eq. (8) into Eq. (11) the Hamiltonian $H_L(t)$ of Eq. (11) can be written as, in the first-order approximation,

$$
\begin{aligned}
H_L(t) = \alpha_0 E + \hbar\omega[a^+(t)a(t) + \frac{1}{2}] + \hbar\Omega_a(t)I_z(t) \\
+2\hbar\Delta k[\Omega_{e0}(t) + \Omega_{e1}(t)]I_z(t)x(t)\sin[\Delta\omega t + \Delta\varphi] \\
+\hbar\Omega_0(t)\{\exp(-i\varphi_0)I^+(0)\exp[ik_0x(t)]\exp[i(\omega_a - \omega_0)t] \\
+\exp(i\varphi_0)\exp[-ik_0x(t)]I^-(0)\exp[-i(\omega_a - \omega_0)t]\} \\
+\hbar\Omega_1(t)\{\exp(-i\varphi_1)I^+(0)\exp[ik_1x(t)]\exp[i(\omega_a - \omega_1)t] \\
+\exp(i\varphi_1)\exp[-ik_1x(t)]I^-(0)\exp[-i(\omega_a - \omega_1)t]\} \qquad (17)
\end{aligned}
$$

where the expansion $\exp[\pm i\Delta kx(t)] = 1 \pm i\Delta kx(t) + ...$ in the Lamb-Dicke limit has been used, the parameter $\Omega_a(t)$ is given by

$$\Omega_a(t) = \omega_a + \frac{4\Omega_0(t)^2}{(\omega_a - \omega_0)} + \frac{4\Omega_1(t)^2}{(\omega_a - \omega_1)} + 2[\Omega_{e0}(t) + \Omega_{e1}(t)]\cos(\Delta\omega t + \Delta\varphi),$$

and the effective Rabi frequency $\Omega_{el}(t) = 2\Omega_0(t)\Omega_1(t)/(\omega_a - \omega_l)$ $(l = 0, 1)$ which was also obtained in Refs. [36]. It is known that the Lamb-Dicke parameter η is defined through $kx(t) = \eta(a(t)^+ + a(t))$. The Lamb-Dicke parameter $\eta_l = \sqrt{(\hbar^2 k_l^2)/(2m\hbar\omega)}$ with the wave vector value $k_l = \omega_l/c$ $(l = 0, 1)$ may be large if the frequencies ω_0 and ω_1 of the two Raman laser light beams are near the transition frequency ω_a $(\sim 10^{15})$ between the atomic internal states $|g_0\rangle$ and $|e\rangle$. However, the wave vector difference $|\Delta k| = |k_0 - k_1|$ may be very small and its value can be controlled in experiment. For example, $|\Delta k| = |\Delta\omega|/c \sim \omega/c << \omega_a/c$, since the oscillatory frequency of the harmonic oscillator $\omega \sim 10^8$ is much less than the atomic internal-state transition frequency $\omega_a \sim 10^{15}$. In this case the Lamb-Dicke parameter may be very small: $\eta = \sqrt{\hbar^2(\Delta k)^2/(2m\hbar\omega)} << 1$, and hence $||\Delta kx(t)|| << 1$.

It seems that one may directly obtain the first-order effective Hamiltonian from Eq. (17) by omitting those terms of the operators $\{I^{\pm}(0)\}$ on the right-hand side of Eq. (17), but the correct treatment is that one should use the first-order approximation solution of the Heisenberg equations (12) which contains the operators $I_z(t)$, $a(t)^+$, and $a(t)$ of the equations (16c), (16d), and (16e) to reduce the Hamiltonian (17) to the first-order effective Hamiltonian. The Hamiltonian (17) may be divided into two parts, the first part does not contain the operators $\{I^{\pm}(0)\}$, while the second contains. Note that the effective Rabi frequency $\Omega_a(t)$ in the Hamiltonian (17) is close to ω_a when the Rabi frequency $\Omega_l(t)$ $(l = 0, 1)$ is much smaller than $|(\omega_a - \omega_l)|$. Then the part that contains the operators $\{I^{\pm}(0)\}$ is nonsecular, since its components generally contain

oscillatory phase factors $\exp[\pm i(\omega_a - \omega_l)t]$ $(l = 0, 1)$ instead of $\exp[\pm i\omega_l t]$ (see Eq. (17)) and in the rotating frame defined by the Hamiltonian $\omega_l I_z$ or $\omega_a I_z$ they still contain largely oscillatory phase factors $\exp[\pm i\omega_a t]$, $\exp[\pm i\omega_l t]$, etc., instead of the slowly varying phase factors $\exp[\pm i\Delta\omega t]$. Thus, the contribution of the nonsecular part to the off-resonance excitation is usually small. If one chooses the suitable parameters of the Raman laser light beams such that the absolute amplitudes in front of the oscillatory phase factors $\exp[\pm i(\omega_a - \omega_l)t]$ $(l = 0, 1)$ in the nonsecular part are much smaller than $|(\omega_a - \omega_l)|$, then the contribution of the nonsecular part to the off-resonance excitation may be neglected. For convenience to discuss below, here the parameters of the Raman adiabatic laser light beams are chosen suitably such that the first part is much more important than the second and hence the nonsecular part that contains the operators $\{I^{\pm}(0)\}$ may be neglected. Then in this case the Hamiltonian (17) is reduced approximately to the simple form

$$H_L(t) = \hbar\omega(a^+ a + \frac{1}{2}) + \hbar\Omega_a(t) I_z + \hbar[\beta(t) a^+ + \beta(t)^* a] I_z \tag{18}$$

where the unity operator term is omitted and the parameter $\beta(t)$ is given by

$$\beta(t) = -i\sqrt{\frac{2\hbar\omega}{m}} \int_0^t dt' \Omega_{eff}(t') \sin[\Delta\omega t' + \Delta\varphi] \exp(i\omega t')$$

$$+\Delta k \sqrt{\frac{2\hbar}{m\omega}} [\Omega_{e0}(t) + \Omega_{e1}(t)] \sin[\Delta\omega t + \Delta\varphi] \exp(i\omega t).$$

In the parameter $\beta(t)$ the integral containing the parameter $\Omega_{eff}(t)$ usually is not considered [36]. If the two frequencies offsets $\{\omega_a - \omega_l\}$ $(l = 0, 1)$ have the opposite sign to each other such that the parameter $|\Omega_{eff}(t)|$ is large, then the dominating contribution to the parameter $\beta(t)$ may come from the integral containing the parameter $\Omega_{eff}(t)$. The first-order effective Hamiltonian (18) can be used to excite a coherent state of harmonic oscillator. The cross operator $[\beta(t)a^+ + \beta(t)^* a] I_z$ in the Hamiltonian $H_L(t)$ (18) is responsible for the internal-state-selective excitation of a coherent state. This will be interpreted below. In the interaction representation defined by the harmonic oscillator Hamiltonian $H_0 = \hbar\omega(a^+ a + 1/2)$ the Hamiltonian (18) is transformed to [11]

$$H_{LI}(t) = \hbar\Omega_a(t) I_z + i\hbar[z(t) a^+ - z(t)^* a] I_z$$

where the complex number $z(t) = \beta(t) \exp[i\omega t - i\pi/2]$. The propagator of the Hamiltonian $H_{LI}(t)$ in the interaction representation is

$$U_{LI}(t) = \exp[-iI_z \int_0^t dt' \Omega_a(t')] D(\mu(t))$$

where the unitary displacement operator $D(\mu(t))$ is defined as [11]

$$D(\mu(t)) = \exp[\mu(t) a^+ - \mu(t)^* a] = T \exp\{I_z \int_0^t dt' [z(t') a^+ - z(t')^* a]\}.$$

Note that here $\mu(t)$ of the operator $D(\mu(t))$ is really an operator and may be expressed $\mu(t) = \mu_1(t)E + 2I_z\mu_2(t)$ [51], where $\mu_l(t)$ $(l = 1, 2)$ is a complex number, $E = (|e\rangle\langle e| + |g_0\rangle\langle g_0|)$, and $I_z = (|e\rangle\langle e| - |g_0\rangle\langle g_0|)/2$. It is known that the unitary displacement operator $D(z(t))$ ($z(t)$ is a complex number) can excite a standard coherent state from the ground state of a harmonic oscillator [11]. Since the operator $\mu(t)$ acts on only the two atomic internal states $|e\rangle$ and $|g_0\rangle$, the unitary displacement operator $D(\mu(t))$ can excite the coherent state of the atom in the harmonic potential well only when the atom is in the internal-state subspace span by the two internal states $|e\rangle$ and $|g_0\rangle$. This indicates that the unitary propagator $U_{LI}(t)$ can excite the coherent state in an internal-state dependent form and hence is internal-state-selective. In order to see more clearly how the first-order effective Hamiltonian (18) can be used to excite selectively the coherent state one may assume that the parameter $\beta(t)$ is real, which could be achieved by choosing suitably the parameters of the Raman laser light beams. Then the Hamiltonian (18) is written as, with the help of Eqs. (13),

$$H_L(t) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \hbar\Omega_a(t)I_z + f(t)I_z x \tag{19}$$

where the force function $f(t) = \sqrt{2m\hbar\omega}\beta(t)$. The last term on the right-hand side of Eq. (19), which is proportional to the product operator $I_z x$, is responsible for the internal-state-selective excitation for the coherent state. In fact, the cross term, for example, the product operator $I_\mu x$ $(\mu = x, y, z)$ (there are also other cross terms $I_\mu y$ and $I_\mu z$ in three dimensions), reflects the coupling between the atomic center-of-mass and internal motions. In order to excite efficiently the coherent state the varying frequency of the force function $f(t)$ should be close to the oscillatory frequency ω of the harmonic oscillator. Now one may write the unitary propagator $U_L(t)$ for the first-order effective Hamiltonian $H_L(t)$ of Eq. (19) as the form

$$U_L(t) = \exp\{-iI_z\int_0^t dt'\Omega_a(t')\}T\exp\{-\frac{i}{\hbar}\int_0^t dt' H_f(t')\} \tag{20}$$

where the first-order effective Hamiltonian $H_f(t)$ is defined by

$$H_f(t) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + f(t)I_z x \tag{21}$$

It is clear from Eq. (3) that the first-order effective Hamiltonian $H_f(t)$ of Eq. (21) is very similar to the Hamiltonian (3) of a forced harmonic oscillator. The Hamiltonian $H_f(t)$ is responsible for the state-selective excitation of a coherent state, as can be seen below.

It is known that at the initial time of the state-selective trigger pulse [1] the halting-qubit atom may be either in the product state $|\psi_0(x)\rangle|g_1\rangle$ (the atom is in the internal state $|g_1\rangle$ and the ground motional state $|\psi_0(x)\rangle$) or in the product state $|\psi_0(x)\rangle|g_0\rangle$ (or $|\psi_0(x)\rangle|e\rangle$), while the two laser light beams are applied selectively to the two internal states $|g_0\rangle$ and $|e\rangle$ and do not have any effect on

other atomic internal states including the state $|g_1\rangle$. Consider the first case that at the initial time the halting-qubit atom is in the product state $|\psi_0(x)\rangle|g_1\rangle$. The time evolution process for the atom is expressed as

$$|\Psi(x,r,t)\rangle = U(t)|\psi_0(x)\rangle|g_1\rangle = \exp\{-iE_1t/\hbar\}U_L(t)|\psi_0(x)\rangle|g_1\rangle.$$

Since the atomic operator $I_z = (|e\rangle\langle e| - |g_0\rangle\langle g_0|)/2$, one has $I_z|g_1\rangle = 0$. Then it follows from Eq. (19) that $H_L(t)|\psi_0(x)\rangle|g_1\rangle = H_0|\psi_0(x)\rangle|g_1\rangle$. Note that $H_0|\psi_0(x)\rangle = \frac{1}{2}\hbar\omega|\psi_0(x)\rangle$. Thus, the time evolution process is reduced to the form

$$|\Psi(x,r,t)\rangle = \exp[-iE_1t/\hbar]\exp[-i\omega t/2]|\psi_0(x)\rangle|g_1\rangle.$$

Up to a global phase factor the state $|\Psi(x,r,t)\rangle$ is really the initial product state $|\psi_0(x)\rangle|g_1\rangle$. This shows that the Raman laser light beams do not have any significant effect on the initial product state $|\psi_0(x)\rangle|g_1\rangle$. On the other hand, for the second case that at the initial time the halting-qubit atom is in the product state $|\psi_0(x)\rangle|g_0\rangle$ the time evolution process for the atom is expressed as

$$|\Psi(x,r,t)\rangle = \exp\{-iI_z\int_0^t dt'\Omega_a(t')\}T\exp\{-\frac{i}{\hbar}\int_0^t dt'H_f(t')\}|\psi_0(x)\rangle|g_0\rangle.$$

Since $I_z|g_0\rangle = -(1/2)|g_0\rangle$, the wave function is reduced to the form

$$|\Psi(x,r,t)\rangle = \exp\{i\frac{1}{2}\int_0^t dt'\Omega_z(t')\}T\exp\{-\frac{i}{\hbar}\int_0^t dt'H_f^-(t')\}|\psi_0(x)\rangle|g_0\rangle$$

where the Hamiltonian $H_f^-(t)$ is given by

$$H_f^-(t) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 - \frac{1}{2}f(t)x.$$

One sees that the Hamiltonian $H_f^-(t)$ is really the Hamiltonian (3) of a forced harmonic oscillator. Thus, the ground state $|\psi_0(x)\rangle$ will be transferred to a coherent state under the Hamiltonian $H_f^-(t)$ [13, 17]. In an analogous way, one can prove that when the halting-qubit atom is in the product state $|\psi_0(x)\rangle|e\rangle$ at the initial time, the time evolution process is given by

$$|\Psi(x,r,t)\rangle = \exp\{-i\frac{1}{2}\int_0^t dt'\Omega_z(t')\}T\exp\{-\frac{i}{\hbar}\int_0^t dt'H_f^+(t')\}|\psi_0(x)\rangle|e\rangle$$

where $I_z|e\rangle = (1/2)|e\rangle$ has been used and the Hamiltonian $H_f^+(t)$ is given by

$$H_f^+(t) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \frac{1}{2}f(t)x.$$

The Hamiltonian $H_f^+(t)$ is also the Hamiltonian (3) of a forced harmonic oscillator and hence the initial ground state $|\psi_0(x)\rangle$ will be excited to a coherent state under the Hamiltonian $H_f^+(t)$ [17]. Here gives a summary for the above

discussion for the off-resonance excitation process. The product state of the atom will keep unchanged under the Raman laser light beams when the atom is in the product state $|\psi_0(x)\rangle|g_1\rangle$ at the initial time. When the atom is changed to the internal state $|g_0\rangle$ or $|e\rangle$ from the original internal state $|g_1\rangle$ and hence in the product state $|\psi_0(x)\rangle|g_0\rangle$ or $|\psi_0(x)\rangle|e\rangle$ at the starting time of the excitation process, the atom will be excited to a coherent state by the Raman laser light beams and moreover the atomic internal state keeps unchanged during the excitation process. Therefore, this coherent-state excitation process is clearly a state-selective excitation process. Such a pulse sequence consisting of the two Raman laser light beams could act as the state-selective trigger pulse to realize the reversible and unitary halting protocol.

The off-resonance selective excitation using the conventional Raman adiabatic laser light beams is a simple technique to prepare approximately a coherent state of an atom or atomic ion in the harmonic potential field. The above theoretic investigation for the off-resonance selective excitation shows ones clearly the mechanism for a Raman adiabatic pulse sequence to excite selectively the coherent state of an atom in the harmonic potential field. The similar theoretical analysis also can be seen in a number of references [19, 20, 25a, 31, 33, 45]. The state-selective excitation process is indeed involved in the coupling between the atomic center-of-mass motion and the internal electronic (or spin) motion of the atom, since there is the cross term xI_z in the effective Hamiltonian $H_L(t)$ of Eq. (19) which is responsible for the state-selective excitation of the coherent state. The coupling between the atomic center-of-mass and internal motions is also a general mechanism for the atomic laser cooling [22] and the decelerating and accelerating processes based on the STIRAP method of a free atom [8]. However, if one compares the present theoretic treatment for an atom in the harmonic potential well to that one for a free atom [8], one can see their difference is quite large. The reason for this is that a motional state of the atom in the harmonic potential field is discrete instead of continuous. Though the ground motional state of the atom in the harmonic potential well has a Gaussian shape, the excited motional state which is generated from the ground motional state by the Raman pulse sequence could not have a Gaussian shape and tends to have a wave-packet shape quite different from a Gaussian shape. For the off-resonance selective excitation above the excited motional state has a Gaussian shape in the first-order approximation, but in a general case it does not have a Gaussian shape and its shape could be complicated. Thus, in this sense the off-resonance selective excitation mentioned above generally is not an optimal technique to construct the state-selective trigger pulse. But it could be considered as a starting point to develop further a better technique to generate state-selectively the standard coherent state of harmonic oscillator. One could exploit the average Hamiltonian theory [37, 39, 54] or the numerical optimization method [38] based on the Heisenberg equations (12) to improve the off-resonance excitation technique. The state-selective excitation pulse obtained with these optimal methods could be more useful in practice. Of course, it must be pointed out that a state-selective excitation pulse is not always equal to a state-selective trigger pulse.

2.3. The coherent average method

Here suggests the programming operator composition method to construct the state-selective trigger pulse in theory. This method, which has been used extensively to construct NMR multi-pulse sequences, is also called the coherent average method in nuclear magnetic resonance spectroscopy [54]. The theoretical basis of the method is the famous Baker-Campbell-Hausdorff (BCH) formula [39, 41] and the Trotter-Suzuki formalism [40, 42]. The BCH formula is also the theoretical basis of the average Hamiltonian theory [37, 54]. It is necessary to show that the state-selective trigger pulse can be constructed theoretically in an error as small as pleased so as to show that both the reversible and unitary halting protocol that is insensitive to its input state and the efficient quantum search process are feasible. Here consider the double-frequency selective excitation method, which may be different from the off-resonance selective excitation method based on the Raman adiabatic laser light beams in the previous subsection. The method uses simply two conventional amplitude- and phase-modulation laser light beams (the plane wave electromagnetic fields), which may not be Raman adiabatic laser light beams, to excite selectively the atomic internal states so as to set up the coupling between the center-of-mass and the internal motion of the atom. In order to treat conveniently the double-frequency state-selective excitation process in the three-level atom system here consider only the special case that the two laser light beams have the specific parameter settings as given below. It is known the electric dipole interaction between the atom and a pair of laser light beams is given by Eq. (8). Now suppose that the two laser light beams are amplitude- and phase-modulating such that their Rabi frequencies and phases satisfy the following match condition:

$$\Omega_0(t) = \Omega_1(t),\ \varphi_0(t) = \alpha + \gamma,\ \varphi_1(t) = (\omega_0 - \omega_1)t - \alpha + \gamma. \tag{22}$$

Then by using the match condition (22) the electric dipole interaction of Eq. (8) is reduced to the form

$$H_1(t) = 2\hbar\Omega_0(t)\exp(-i\omega_0 t)I^+\exp[i\frac{1}{2}(k_0+k_1)x - i\gamma]\cos[\frac{1}{2}(k_0-k_1)x - \alpha]$$

$$+2\hbar\Omega_0(t)\exp(i\omega_0 t)I^-\exp[-i\frac{1}{2}(k_0+k_1)x + i\gamma]\cos[\frac{1}{2}(k_0-k_1)x - \alpha] \tag{23}$$

where the phases α and γ can be set suitably in experiment. The theoretical treatment could become more convenient for the state-selective excitation process when the electric dipole interaction $H_1(t)$ takes the form of Eq. (23), since the electric dipole interaction (23) is modulated by a single frequency ω_0. On the other hand, it could be convenient to treat the state-selective excitation process in the rotating reference frame. It is known that the Schrödinger equation for a quantum system in the interaction representation can be written as [2]

$$i\hbar\frac{\partial}{\partial t}\Psi_I(x,t) = H_I(t)\Psi_I(x,t)$$

where the state $\Psi_I(x,t)$ of the quantum system in the interaction representation is related to the state $\Psi_L(x,t)$ in the laboratory frame by the unitary transformation: $\Psi_I(x,t) = U_0(t)^+ \Psi_L(x,t)$, and the Hamiltonian $H_I(t)$ of the quantum system in the interaction representation to the Hamiltonian $H(t) = H_0(t) + H_1(t)$ in the laboratory frame by

$$H_I(t) = U_0(t)^+ H_1(t) U_0(t) \tag{24}$$

where the unitary propagator $U_0(t)$ is defined as

$$U_0(t) = T \exp\{-\frac{i}{\hbar}\int_0^t dt' H_0(t')\}. \tag{25}$$

The total unitary propagator of the quantum system then is written as

$$U(t) = T \exp\{-\frac{i}{\hbar}\int_0^t dt' [H_0(t') + H_1(t')]\} = U_0(t) U_I(t) \tag{26}$$

where the unitary propagator of the quantum system in the interaction representation is defined as

$$U_I(t) = T \exp\{-\frac{i}{\hbar}\int_0^t dt' H_I(t')\}. \tag{27}$$

One may choose a suitable rotating reference frame or the interaction representation for convenient treatment of the state-selective excitation process. First of all, the atomic internal Hamiltonian of Eq. (4) is rewritten as $H(r) = \alpha_0 E + \hbar\omega_a I_z + (E_1 - \alpha_0)|g_1\rangle\langle g_1|$, where E is the 3×3 unity matrix. Then by neglecting the unity operator the total Hamiltonian (1) for the three-state atom system is rewritten as

$$H(t) = H_0 + \hbar\omega_a I_z + (E_1 - \alpha_0)(|g_1\rangle\langle g_1|) + H_1(t) \tag{28}$$

where H_0 and $H_1(t)$ are given by Eqs. (2) and (23), respectively. Now the rotating reference frame may be defined by the atomic internal Hamiltonian $H_0(r) = \hbar\omega_0 I_z + (E_1 - \alpha_0)(|g_1\rangle\langle g_1|)$. In the rotating frame the wave function is

$$\Psi_I(x,t) = \exp[i(E_1 - \alpha_0)(|g_1\rangle\langle g_1|)t/\hbar] \exp(i\omega_0 I_z t)\Psi_L(x,t),$$

and the total Hamiltonian $H(t)$ of Eq. (28) of the atom system is replaced with the Hamiltonian $H_I(t)$:

$$H_I(t) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \hbar(\omega_a - \omega_0)I_z$$

$$+2\hbar\Omega_0(t) I^+ \exp[i\frac{1}{2}(k_0 + k_1)x - i\gamma] \cos[\frac{1}{2}(k_0 - k_1)x - \alpha]$$

$$+2\hbar\Omega_0(t) I^- \exp[-i\frac{1}{2}(k_0 + k_1)x + i\gamma] \cos[\frac{1}{2}(k_0 - k_1)x - \alpha] \tag{29}$$

where the electric dipole interaction (23) has been used. In the following consider a simple case that the on-resonance condition is met, that is, $(\omega_a - \omega_0) = 0$, and the amplitude $\Omega_0(t)$ is time-independent, that is, $\Omega_0(t) = \Omega_0$. Then the Hamiltonian $H_I(t)$ of Eq. (29) is reduced to the time-independent form

$$H_I(t) \equiv H_I = H_0 + H_I(\alpha, \gamma) \tag{30}$$

where H_0 is still given by Eq. (2) and the electric dipole interaction $H_I(\alpha, \gamma)$ is written as

$$H_I(\alpha, \gamma) = 2\hbar\Omega_0 I^+ \exp[i\frac{1}{2}(k_0 + k_1)x - i\gamma] \cos[\frac{1}{2}(k_0 - k_1)x - \alpha]$$

$$+2\hbar\Omega_0 I^- \exp[-i\frac{1}{2}(k_0 + k_1)x + i\gamma] \cos[\frac{1}{2}(k_0 - k_1)x - \alpha]. \tag{31}$$

Obviously, the Hamiltonian $H_I(\alpha, \gamma)$ is dependent upon the phases α and γ. One may take the time-independent Hamiltonian H_I of Eq. (30) as the basic Hamiltonian to construct the state-selective trigger pulse. The unitary propagator of the atom system corresponding to the Hamiltonian (30) is written as

$$U_I(t) = \exp[-iH_I t/\hbar]. \tag{32}$$

On the other hand, in the absence of the two laser light beams the time evolution process of the atom system is governed by the Hamiltonian H_0 of Eq. (2) and its unitary propagator is given by

$$U_o(t) = \exp[-iH_0 t/\hbar], \tag{33a}$$

and its inverse propagator by

$$U_o^+(t) = \exp[iH_0 t/\hbar]. \tag{33b}$$

It can turn out in next section that by using the same Hamiltonian H_0 of Eq. (2) of the harmonic oscillator one can generate the inverse unitary propagator $U_o^+(t)$ with any time $t \neq k\pi/\omega$ up to a global phase factor. In fact, there are the unitary operator identities:

$$U_o(t)U_o(t_1) = U_o(t_1)U_o(t) = \exp[i\beta(t_1)]E, \tag{34}$$

where E is the unit operator, $\exp[i\beta(t_1)]$ is a global phase factor, and the time $t_1 = 2k\pi/\omega - t$ $(k = 1, 2, ...)$. It follows from Eqs. (33) and (34) that one may define the inverse unitary operator $U_o^+(t)$ as

$$U_o^+(t) = \exp[-i\beta(t_1)]U_o(t_1). \tag{35}$$

Hereafter the unitary operator $U_o(t_1)$ is also called the inverse operator of the unitary operator $U_o(t)$, although it has a difference of a global phase factor from the real inverse unitary operator $U_o^+(t)$. The unitary propagators $U_o(t)$ with any time t of the harmonic oscillator in the absence of the two laser light

beams can be realized directly in experiment. It follows from Eq. (35) that up to a global phase factor the inverse propagator $U_o^+(t)$ also can be realized in experiment as the unitary propagator $U_o(t_1)$ of the harmonic oscillator can be realized in experiment. On the other hand, the unitary propagator $U_I(t)$ (32) of the harmonic oscillator in the rotating frame also can be realized directly by the two laser light beams. These three realizable unitary propagators $U_o(t)$, $U_o^+(t)$, and $U_I(t)$ in the rotating frame are the basic unitary propagators to construct a general state-selective trigger pulse.

At the first step the unitary propagator of the electric dipole interaction $H_I(\alpha,\gamma)$ is created through the three basic unitary propagators. The unitary propagator of the electric dipole interaction $H_I(\alpha,\gamma)$ is defined by

$$U_I(\alpha,\gamma,t) = \exp[-iH_I(\alpha,\gamma)t/\hbar]. \tag{36}$$

It is known from Eq. (30) that the electric dipole interaction $H_I(\alpha,\gamma)$ may be written as $H_I(\alpha,\gamma) = H_I - H_0$. Thus, one may first carry out in experiment a composite pulse sequence:

$$U_o(t_1)U_I(t) = \exp[i\beta(t_1)]U_o^+(t)U_I(t)$$

$$= \exp[i\beta(t_1)]\exp[iH_0t/\hbar]\exp[-iH_It/\hbar]. \tag{37}$$

The physical meaning for the pulse sequence (37) is that a pair of laser light beams whose parameters satisfy the match condition (22) and time period is t are first applied to the atom in the harmonic potential well, then the pair of laser light beams are turned off, and the atom then evolves in the time period t_1 in the harmonic potential well without any external laser light field. If the time interval δt is sufficiently short, then the unitary operator $U_o^+(\delta t)U_I(\delta t)$ is approximately equal to the unitary propagator $U_I(\alpha,\gamma,\delta t)$ according to the famous Baker-Campbell-Hausdorff (BCH) formula for a product of exponential operators [39, 40, 41],

$$U_I(\alpha,\gamma,\delta t) = U_o^+(\delta t)U_I(\delta t) + O((\delta t)^2). \tag{38}$$

Since the composite pulse sequence $U_o(t_1)U_I(t)$ in the rotating frame can be realized directly in experiment, the composite unitary operator $U_o^+(\delta t)U_I(\delta t)$ may be realized up to a global phase factor, as shown in Eq. (37). Thus, the unitary propagator $U_I(\alpha,\gamma,\delta t)$ may be realized in experiment up to a global phase factor, as can be seen from Eq. (38). There is an error term $O((\delta t)^2)$ that is proportional to $(\delta t)^2$ between the desired unitary propagator $U_I(\alpha,\gamma,\delta t)$ and the composite unitary operator $U_o^+(\delta t)U_I(\delta t)$ on the right-hand side of Eq. (38). A much better composition for $U_I(\alpha,\gamma,\delta t)$ with an arbitrary higher-order approximation can also be obtained from the two propagators $U_o^+(\delta t)$ and $U(\delta t)$ by using the Trotter-Suzuki formalism [40, 42]. For example, according to the Trotter-Suzuki formalism [42] one can obtain a better symmetric composition for the unitary operator $U_I(\alpha,\gamma,\delta t)$ by the multi-pulse sequence:

$$S_1(\delta t) = U_o^+(\delta t/2)U_I(\delta t)U_o^+(\delta t/2), \tag{38a}$$

$$U_I(\alpha,\gamma,\delta t) = S_1(\delta t) + O((\delta t)^3),$$

or a much better symmetric composition by the multi-pulse sequence:

$$S_{2n-1}(\delta t) = [S_{2n-3}(p_n\delta t)]^2 S_{2n-3}((1-4p_n)\delta t)[S_{2n-3}(p_n\delta t)]^2, \tag{38b}$$

$$U_I(\alpha,\gamma,\delta t) = S_{2n-1}(\delta t) + O((\delta t)^{2n+1}),$$

where $n \geq 2$ and $p_n = (4 - 4^{1/(2n-1)})^{-1}$. Thus, without losing generality here suppose that the unitary operator $U_I(\alpha,\gamma,\delta t)$ can be constructed as exactly as pleased. The unitary operator $U_I(\alpha,\gamma,\delta t)$ will be further used to build up the state-selective trigger pulse.

Now examine the commutation relation between the electric-dipole interactions $H_I(\alpha,\gamma)$ in two different phases $\gamma = 0$ and $\pi/2$. It can turn out from the electric dipole interaction of Eq. (31) that the hermite commutation operator Q takes the form

$$Q \equiv i[H_I(\alpha,0), H_I(\alpha,\pi/2)] = -16\hbar^2\Omega_0^2 I_z \cos^2[\frac{1}{2}(k_0-k_1)x - \alpha] \tag{39}$$

where the commutation relations $2I_z = [I^+, I^-]$ and $[I^+, I^+] = [I^-, I^-] = 0$ have been used. If the phase $\alpha = \pi/4$ and the Lamb-Dicke limit is met, that is, $||(k_0-k_1)x|| << 1$, then the operator $\cos^2[\frac{1}{2}(k_0-k_1)x-\alpha]$ can be expanded as

$$\cos^2[\frac{1}{2}(k_0-k_1)x-\alpha] = \frac{1}{2} + \frac{1}{2}(k_0-k_1)x + O(||(k_0-k_1)x||^3).$$

Thus, the hermitian operator Q can be written as

$$Q = -8\hbar^2\Omega_0^2 I_z - 8\hbar^2(k_0-k_1)\Omega_0^2 I_z x + O(||(k_0-k_1)x||^3). \tag{40}$$

Obviously, the second term on the right-hand side of the operator Q is the cross term $I_z x$ and is responsible for the state-selective excitation of the coherent state. Therefore, at the second step one should construct the unitary propagator $\exp(i\lambda Q)$. The unitary operator $\exp(i\lambda Q)$ can be generated from the unitary operators $\{U_I(\alpha,\gamma,\delta t)\}$ with the help of the BCH formula [41] and it is realized by the multi-pulse sequence:

$$U_I(\alpha,0,\delta t)U_I(\alpha,\pi/2,\delta t)U_I(\alpha,0,\delta t)^+ U_I(\alpha,\pi/2,\delta t)^+$$

$$= \exp\{-[H_I(\alpha,0), H_I(\alpha,\pi/2)](\delta t/\hbar)^2\} + O((\delta t)^3). \tag{41a}$$

A better composition for the unitary operator $\exp(i\lambda Q)$ can also be obtained, for example, the following multi-pulse sequence will lead to a better result,

$$\exp[iQ(\delta t/\hbar)^2] = U_I(\alpha,0,\delta t/\sqrt{2})U_I(\alpha,\pi/2,\delta t/\sqrt{2})$$

$$\times[U_I(\alpha,0,\delta t/\sqrt{2})^+ U_I(\alpha,\pi/2,\delta t/\sqrt{2})^+]^2$$

$$\times U_I(\alpha,0,\delta t/\sqrt{2})U_I(\alpha,\pi/2,\delta t/\sqrt{2}) + O((\delta t)^4). \tag{41b}$$

Furthermore, it is possible to obtain a much better composition for the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$ by using the Trotter-Suzuki formalism [42, 43]. Here it will not be further discussed in detail. By neglecting the error term $O(||(k_0 - k_1)x||^3)$ on the right-hand side of Eq. (40) the unitary operator $\exp[iQ(\delta t/\hbar)^2]$ is written as

$$\exp[iQ(\delta t/\hbar)^2] = \exp\{-i8\Omega_0^2[I_z + \Delta k I_z x](\delta t)^2\}. \tag{42}$$

It can turn out below that the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$ can excite directly and state-selectively the ground motional state of the harmonic oscillator to a Gaussian wave-packet motional state with a high motional energy.

Now suppose that at the initial time the atom is in the product state $\exp(i\varphi_0) \times|\psi_0(x)\rangle|g_1\rangle$ in the rotating frame, where $\exp(i\varphi_0)$ is a global phase factor. Applying the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$ to the initial product state one obtains

$$|\Psi(x,r,t)\rangle = \exp[iQ(\delta t/\hbar)^2]\exp(i\varphi_0)|\psi_0(x)\rangle|g_1\rangle = \exp(i\varphi_0)|\psi_0(x)\rangle|g_1\rangle \tag{43}$$

where the eigen-equation $I_z|g_1\rangle = 0|g_1\rangle$ has been used. Thus, the initial product state keeps unchanged under the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$. On the other hand, if the initial product state of the atom is $\exp(i\varphi_0)|\psi_0(x)\rangle|g_0\rangle$ or $\exp(i\varphi_0)|\psi_0(x)\rangle|e\rangle$ in the rotating frame, then applying the unitary propagator to the initial product state one obtains the atomic product state

$$|\Psi(x,r,t)\rangle = \exp(i\varphi_0)\exp\{i4\Omega_0^2(\delta t)^2\}\exp\{i4\Omega_0^2\Delta k(\delta t)^2 x\}|\psi_0(x)\rangle|g_0\rangle. \tag{43a}$$

or

$$|\Psi(x,r,t)\rangle = \exp(i\varphi_0)\exp\{-i4\Omega_0^2(\delta t)^2\}\exp\{-i4\Omega_0^2\Delta k(\delta t)^2 x\}|\psi_0(x)\rangle|e\rangle. \tag{43b}$$

where the eigen-equations $I_z|g_0\rangle = (-1/2)|g_0\rangle$ and $I_z|e\rangle = (1/2)|e\rangle$ have been used. If now the rotating frame is changed back to the laboratory frame, then only a global phase factor is generated for each of the three atomic product states $|\Psi(x,r,t)\rangle$ of Eqs. (43), (43a), and (43b). Thus, the atomic product states $|\Psi(x,r,t)\rangle$ of Eqs. (43), (43a), and (43b) are really the final atomic product states after the atom is applied by the double-frequency pulse sequence that generates the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$. Now the ground motional state $|\psi_0(x)\rangle$ of the harmonic oscillator takes the Gaussian form

$$|\psi_0(x)\rangle = [\frac{1}{2\pi(\Delta x)_0^2}]^{1/4}\exp[-\frac{1}{4}\frac{x^2}{(\Delta x)_0^2}] \tag{44}$$

where $(\Delta x)_0^2 = (\frac{\hbar}{2m\omega})$ and ω is the oscillatory frequency of the harmonic oscillator. Obviously, this motional state keeps unchanged during the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$ acting on the atom when the atom is in the internal state $|g_1\rangle$, as shown in Eq. (43). However, the atomic product states (43a) and (43b) show that after the unitary operator $\exp[iQ(\delta t/\hbar)^2]$ acts on the initial product state $\exp(i\varphi_0)|\psi_0(x)\rangle|g_0\rangle$ or $\exp(i\varphi_0)|\psi_0(x)\rangle|e\rangle$ the atom is in the

motional state

$$|\Psi(x,t)\rangle = \exp[i\varphi(t)][\frac{1}{2\pi(\Delta x)_0^2}]^{1/4}\exp[-\frac{1}{4}\frac{x^2}{(\Delta x)_0^2}]\exp[ip_0x/\hbar] \tag{44a}$$

or

$$|\Psi(x,t)\rangle = \exp[i\phi(t)][\frac{1}{2\pi(\Delta x)_0^2}]^{1/4}\exp[-\frac{1}{4}\frac{x^2}{(\Delta x)_0^2}]\exp[-ip_0x/\hbar] \tag{44b}$$

where the global phase factor $\exp[i\varphi(t)] = \exp(i\varphi_0)\exp\{i4\Omega_0^2(\delta t)^2\}$, $\exp[i\phi(t)] = \exp(i\varphi_0)\exp\{-i4\Omega_0^2(\delta t)^2\}$, and the mean momentum p_0 is given by $p_0/\hbar = 4\Omega_0^2\Delta k(\delta t)^2$. Both the motional states (44a) and (44b) are the Gaussian wave-packet states with the center-of-mass positions $x_0 = 0$ and the momentums p_0 and $-p_0$, respectively. Thus, the two motional states show that after the double-frequency pulse sequence is turned off the atom is still in the original harmonic potential well and has approximately the mean motional energy,

$$E_0 = p_0^2/(2m) + \frac{1}{2}m\omega^2x_0^2 = [4\hbar\Delta k\Omega_0^2(\delta t)^2]^2/(2m). \tag{45}$$

This motional energy E_0 can be much larger than the zero-point energy of the atom in the harmonic potential well. On the other hand, the motional state (44a) shows that at the end of the excitation process of the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$ the atom in the internal state $|g_0\rangle$ moves along the direction $+x$ with the velocity p_0/m in the harmonic potential well, while the motional state (44b) shows that the atom in the internal state $|e\rangle$ moves along the direction $-x$ with the same velocity p_0/m. The above investigation for the effect of the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$ on the initial product state of the atom is summarized as follows. When the atom is in the internal state $|g_1\rangle$ at the initial time, the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$ does not have any significant effect on the initial product state. In particular, even if the initial motional state of the atom is an arbitrary wave function, i.e., a superposition motional state, the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$ does not yet have any siginificant effect on the motional state when the atom is in the internal state $|g_1\rangle$ at the initial time. However, the initial ground motional state of the atom will be transferred to a Gaussian wave-packet state with a higher motional energy by the unitary propagator if the atom is in the internal state $|g_0\rangle$ or $|e\rangle$ at the initial time. Therefore, the double-frequency pulse sequences (41) to generate the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$ may generally act as the state-selective trigger pulse in the quantum control process [1].

In order to implement the double-frequency pulse sequences (41) one needs to realize not only the unitary propagators $\{U_I(\alpha,\gamma,t)\}$ but also their inverse propagators $\{U_I^+(\alpha,\gamma,t)\}$. The inverse unitary operators $\{U_I^+(\alpha,\gamma,\delta t)\}$ can be generated as follows. Since the electric dipole Hamiltonian $H_I(\alpha,\gamma)$ of Eq. (31) is time-independent, these inverse unitary operators may be generated by inverting the electric dipole Hamiltonian: $H_I(\alpha,\gamma) \rightarrow -H_I(\alpha,\gamma)$. Since the electric dipole Hamiltonian is dependent upon the phases α and

γ, it is possible to choose suitably the phase γ to obtain the negative-sign Hamiltonian $-H_I(\alpha,\gamma)$. In fact, it follows from Eq. (31) that $H_I(\alpha,0) = -H_I(\alpha,\pi)$ and $H_I(\alpha,\pi/2) = -H_I(\alpha,3\pi/2)$. Therefore, the inverse unitary operators $U_I^+(\alpha,0,\delta t)$ and $U_I^+(\alpha,\pi/2,\delta t)$ are respectively given by

$$U_I^+(\alpha,0,\delta t) = U_I(\alpha,\pi,\delta t),\ U_I^+(\alpha,\pi/2,\delta t) = U_I(\alpha,3\pi/2,\delta t). \tag{46}$$

Note that the unitary propagator $U_I(\alpha,\gamma,\delta t)$ for any phase values α and γ can be implemented in experiment. Then the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$ can be implemented in experiment, since it follows from Eqs. (41b) and (46) that the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$ can be expressed as

$$\begin{aligned}\exp[iQ(\delta t/\hbar)^2] &= U_I(\alpha,0,\delta t/\sqrt{2})U_I(\alpha,\pi/2,\delta t/\sqrt{2})U_I(\alpha,\pi,\delta t/\sqrt{2})\\ &\times U_I(\alpha,3\pi/2,\delta t/\sqrt{2})U_I(\alpha,\pi,\delta t/\sqrt{2})U_I(\alpha,3\pi/2,\delta t/\sqrt{2})\\ &\times U_I(\alpha,0,\delta t/\sqrt{2})U_I(\alpha,\pi/2,\delta t/\sqrt{2}) + O((\delta t)^4).\end{aligned} \tag{47}$$

The double-frequency pulse sequence (47) of the propagator $\exp[iQ(\delta t/\hbar)^2]$ consists of a number of the three realizable unitary propagators $U_o(t)$, $U_o^+(t)$, and $U_I(t)$. Obviously, it can be efficiently implemented in experiment, as can be seen from Eqs. (37), (38), (38a), and (47). It should be pointed out that one also can obtain a much better multi-pulse sequence than the sequence (47) for the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$ with the help of the Trotter-Suzuki formalism [42, 43]. This fact tells ones that the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$ can be implemented efficiently and as exactly as pleased.

On the other hand, the internal-state rotating operator $R_z(\theta) = \exp(-i\theta I_z)$ of the atom in the harmonic potential well in the rotating frame can also be prepared in a similar way to generating the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$. When the phase $\alpha = 0$ and the Lamb-Dicke limit $||(k_0 - k_1)x|| << 1$ is met, the hermitian operator Q of Eq. (39) may be written as

$$Q \equiv -16\hbar^2\Omega_0^2 I_z + O(||(k_0 - k_1)x||^2).$$

If the error term $O(||(k_0 - k_1)x||^2)$ is neglected, then the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$ is really the internal-state rotating operator:

$$R_z(\theta) = \exp[iQ(\delta t/\hbar)^2] = \exp\{-i16\Omega_0^2(\delta t)^2 I_z\} \tag{48}$$

where the rotating angle $\theta = 16\Omega_0^2(\delta t)^2$. This rotating operator is independent of any atomic motional state but applied only to the two atomic internal states $|g_0\rangle$ and $|e\rangle$ selectively. Here it must be pointed out that the inverse propagator of the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$ and the inverse operator of the internal-state rotating unitary operator $R_z(\theta)$ can also be implemented in experiment, as can be seen from Eqs. (41) and (47), because both the unitary operator $U_I(\alpha,\gamma,t)$ and its inverse operator $U_I^+(\alpha,\gamma,t)$ can be implemented in experiment.

In this section the on-resonance condition $\omega_a = \omega_0$ has been used in the Hamiltonian (29) to simplify the construction of the propagator $\exp[iQ(\delta t/\hbar)^2]$.

For a general case that the on-resonance condition does not hold, that is, $\omega_a \neq \omega_0$, one may use π pulses $\exp(-i\pi I_x)$ and/or $\exp(-i\pi I_y)$ to refocus the term $\hbar(\omega_a - \omega_0)I_z$ in the Hamiltonian (29), where the π pulses $\exp(-i\pi I_x)$ and $\exp(-i\pi I_y)$ may be generated by an ultrashort laser light pulse. Then in the general case the coherent average method can be used as well to build up the unitary propagator $\exp[iQ(\delta t/\hbar)^2]$. The programming operator composition method has also been used to build up the quantum gates and/or the internal-state-selective quantum gates in the trapped ion systems [44, 45, 46].

3. Manipulating the complex linewidth of a Gaussian wave-packet state

The Gaussian wave-packet motional state or the standard coherent state generated by the state-selective trigger pulse must have a higher motional energy than the ground state of the harmonic oscillator, but the Gaussian wave-packet state generated by the state-selective trigger pulse could not have an expected complex linewidth. However, in the quantum control process to realize the reversible and unitary state-insensitive halting protocol and the efficient quantum search process it could be required that the Gaussian wave-packet state of the halting-qubit atom have an adjustable complex linewidth after the state-selective trigger pulse really acts on the atom. Thus, one needs to construct a pulse sequence to control the complex linewidth of a Gaussian wave-packet state. This pulse sequence combining with the state-selective trigger pulse will form a composite state-selective trigger pulse. This composite state-selective trigger pulse may manipulate not only the center-of-mass position and momentum but also the complex linewidth of the Gaussian wave-packet state when the halting-qubit atom is really acted on by the composite state-selective trigger pulse. Below it is discussed how to generate a pulse sequence to manipulate the complex linewidth of a Gaussian wave-packet state. The complex linewidth for a Gaussian wave-packet state could be controlled by an external harmonic potential field. It is known that the initial motional state of the halting-qubit atom is prepared to be the ground state of the harmonic oscillator, which is the Gaussian wave-packet state $|\psi_0(x)\rangle$ of Eq. (44), and the harmonic potential field that is applied to the atom has the oscillatory frequency ω. Now this harmonic potential field is switched to another harmonic potential field with the oscillatory frequency ω_c at the initial time. The atom then undergoes a time evolution process in the new harmonic potential field. This evolution process may be described by the unitary propagator of the harmonic oscillator with the oscillatory frequency ω_c, which in the coordinate representation may be expressed as [13, 14, 21]

$$G(x_b, t_b; x_a, t_a) = \sqrt{\frac{m\omega_c}{i2\pi\hbar}} \exp[-i\frac{m\omega_c}{\hbar}x_a x_b] \tag{49}$$

where the period of the process $T_c = t_b - t_a$ is chosen such that it satisfies $\cos[\omega_c T_c] = 0$ and $\sin[\omega_c T_c] = 1$, that is, $T_c = [2k\pi + \pi/2]/\omega_c$ $(k = 0, 1, ...)$. The

atomic wave function at the end of the process is calculated by

$$\Psi(x_b,t_b)=\int dx_a G(x_b,t_b;x_a,t_a)\psi_0(x_a)$$

$$=[\frac{1}{2\pi(\Delta x)_0^2}]^{1/4}\sqrt{\frac{m\omega_c}{i2\pi\hbar}}\int dx_a\exp(-ax_a^2+bx_a),$$

where the ground state $\psi_0(x_a)$ of Eq. (44) is used and the parameter $a=[4(\Delta x)_0^2]^{-1}$ and $b=-i\hbar^{-1}m\omega_c x_b$. By using the Gaussian integral formula:

$$\int dx_a\exp(-ax_a^2+bx_a)=\sqrt{\frac{\pi}{a}}\exp(\frac{b^2}{4a}) \tag{50}$$

one obtains the wave function:

$$\Psi(x_b,t_b)=\exp(i\phi_0)[\frac{1}{2\pi(\Delta x)^2}]^{1/4}\exp\{-\frac{1}{4}\frac{x_b^2}{(\Delta x)^2}\}, \tag{51}$$

where the phase $\phi_0=-\pi/4$ and the wave-packet spreading is just $\varepsilon=\sqrt{2}(\Delta x)$, and $(\Delta x)^2$ is given by

$$(\Delta x)^2=[\frac{\hbar}{2m\omega_c(\Delta x)_0}]^2=(\frac{\omega}{\omega_c})(\frac{\hbar}{2m\omega_c})$$

where $(\Delta x)_0^2=(\frac{\hbar}{2m\omega})$ is used. The imaginary part of the complex linewidth of the Gaussian wave-packet state (51) is zero and the real part is $(\Delta x)^2$. Therefore, the wave-packet spreading ε or the real part of the complex linewidth is controlled by both the oscillatory frequencies ω_c and ω of the harmonic potential fields. The initial wave-packet motional state $\psi_0(x)$ (44) usually may be prepared to have a small and fixed wave-packet spreading $\varepsilon_0=\sqrt{2}(\Delta x)_0$ which corresponds to a large oscillatory frequency ω. For example, if one takes $\omega\sim 10^8$, then $(\Delta x)_0^2\sim 10^{-17}$ and $\omega\hbar/(2m)\sim 0.1$ for the atomic mass $m\sim 10^{-25}$Kg. Then by setting suitably the oscillatory frequency ω_c one may obtain the desired wave-packet spreading or the real part of the complex linewidth for the state $\Psi(x_b,t_b)$ (51). The important thing is that the center-of-mass position and momentum for the Gaussian wave-packet state of the atom keeps unchanged in the evolution process, as can be seen from the state $\psi_0(x)$ (44) and the state $\Psi(x_b,t_b)$ (51). This means that after the evolution process the atom is still in the original position $x=0$ in the coordinate axis and has zero momentum. One therefore concludes that the real part of the complex linewidth of a Gaussian wave-packet state of an atom may be manipulated by varying the oscillatory frequency of the harmonic potential field applying to the atom.

The imaginary part of the complex linewidth of a Gaussian wave-packet state may also be manipulated at will. One of the simplest and most intuitive methods to manipulate the imaginary part of the complex linewidth is that the atom undergoes simply a free-particle motion or an inverse free-particle motion. It is well known that the wave-packet spreading of the Gaussian wave-packet

state of a free atom (the initial imaginary part of the complex linewidth is zero) becomes larger and larger when the atom undergoes a free-particle motion [2]. Obviously, the imaginary part of the complex linewidth will become less and less if the atom undergoes the inverse free-particle motion. Suppose that at the initial time the atom in the harmonic potential well is in the Gaussian wave-packet state of Eq. (51). Now one turns off the harmonic potential field applying to the atom. Then the atom becomes a free atom. However, the atom does not leave its original position even after the harmonic potential field is switched off, since the atomic motional momentum is zero before the harmonic potential field is turned off. Therefore, the only change for the Gaussian wave-packet state (51) of the free atom is its complex linewidth after the harmonic potential field is turned off. The time evolution process of the atom after the harmonic potential field is turned off may be calculated by using the free-particle unitary propagator. It is known that the unitary propagator of a free particle is given by [13, 14, 21]

$$G(x_b, t_b; x_a, t_a) = \sqrt{\frac{m}{i2\pi hT}} \exp[i\frac{m}{2\hbar T}(x_b - x_a)^2] \tag{52a}$$

where $T = t_b - t_a$ is the time period of the free-particle motion. After the harmonic potential field is turned off the free atom undergoes a free-particle motion with the time period T and its motional state $\Psi(x_a, t_a)$ of Eq. (51) is changed to another Gaussian wave-packet state:

$$\Psi_f(x_b, t_b) = \exp(i\phi_0)[\frac{1}{2\pi(\Delta x)^2}]^{1/4} \exp\{-\frac{1}{4}\frac{x_b^2}{(\Delta x)^2 + i(\frac{\hbar T}{2m})}\}. \tag{53a}$$

Here the complex linewidth of the Gaussian wave-packet state $\Psi_f(x_b, t_b)$ is given by $W(T) = (\Delta x)^2 + i\hbar T/(2m)$. Thus, the imaginary part of the complex linewidth is proportional to the time period T of the free-particle motion. However, the imaginary part $\hbar T/(2m)$ is always positive. In order to achieve a negative imaginary part one may let the atom perform an inverse free-particle motion. The unitary propagator for the inverse free-particle motion may be given by

$$G^+(x_b, t_b; x_a, t_a) = \sqrt{-\frac{m}{i2\pi\hbar T}} \exp[-i\frac{m}{2\hbar T}(x_b - x_a)^2] \tag{52b}$$

The unitary propagator of the inverse free-particle motion may be generated with the help of the external quadratic potential field (see below). Now the state $\Psi(x_a, t_a)$ of Eq. (51) is changed to the Gaussian wave-packet state $\Psi_i(x_b, t_b)$ after the inverse free-particle motion,

$$\Psi_i(x_b, t_b) = \exp[i\varphi_0][\frac{1}{2\pi(\Delta x)^2}]^{1/4} \exp\{-\frac{1}{4}\frac{x_b^2}{(\Delta x)^2 - i(\frac{\hbar T}{2m})}\}. \tag{53b}$$

Here the complex linewidth of the state $\Psi_i(x_b, t_b)$ is given by $W = (\Delta x)^2 - i\hbar T/(2m)$. Its imaginary part is negative. Thus, the imaginary part of the

complex linewidth can be controlled by the free-particle motion and the inverse free-particle motion. Obviously, due to the fact that the atomic motional momentum is zero the center-of-mass position ($x_0 = 0$) of the Gaussian wave-packet state (51) keeps unchanged in the free-particle motion and the inverse free-particle motion. The manipulating method by using the free-particle motion and the inverse free-particle motion is very simple. However, there could be a disadvantage for the manipulation: if a large imaginary part of the complex linewidth needs to be achieved, then one needs to spend a long time for it.

In the preceding discussion one needs to use the inverse unitary propagator of a free particle to manipulate the imaginary part of the complex linewidth. Here gives the explicit expression for the inverse unitary propagator without a detail proof. It is known that the unitary propagator of a free particle is given by $U_f(t) = \exp[-\frac{p^2 t}{2m\hbar}]$, p is the momentum operator of the free particle. Denote that $U_{ok}(t_k) = \exp[-iH_{0k}t_k/\hbar]$ $(k = 1, 2)$ is a unitary propagator of a harmonic oscillator. The Hamiltonian H_{0k} of the harmonic oscillator is $H_{0k} = p^2/(2m) + m\omega_k^2 x^2/2$. Then it can prove that the inverse unitary propagator of a free particle can be written as, up to a global phase factor,

$$U_f(T)^+ \equiv \exp[i\frac{p^2 T}{2m\hbar}] = U_{o1}(T_1)U_f(T)U_{o2}(T_2)$$

where the time intervals T_1 and T_2 (or the oscillatory frequencies ω_1 and ω_2 of the two harmonic potential fields) are determined through

$$\sin(\omega_1 T_1) = \mp\frac{2T\omega_1\omega_2^2}{\sqrt{[T^2\omega_1^2\omega_2^2 - (\omega_2^2 - \omega_1^2)]^2 + [2T\omega_1\omega_2^2]^2}},$$

$$\cos(\omega_1 T_1) = \mp\frac{[T^2\omega_1^2\omega_2^2 - (\omega_2^2 - \omega_1^2)]}{\sqrt{[T^2\omega_1^2\omega_2^2 - (\omega_2^2 - \omega_1^2)]^2 + [2T\omega_1\omega_2^2]^2}}$$

and

$$\sin(\omega_2 T_2) = \pm\frac{2T\omega_1^2\omega_2}{\sqrt{[T^2\omega_1^2\omega_2^2 + (\omega_2^2 - \omega_1^2)]^2 + [2T\omega_1^2\omega_2]^2}},$$

$$\cos(\omega_2 T_2) = \pm\frac{[T^2\omega_1^2\omega_2^2 + (\omega_2^2 - \omega_1^2)]}{\sqrt{[T^2\omega_1^2\omega_2^2 + (\omega_2^2 - \omega_1^2)]^2 + [2T\omega_1^2\omega_2]^2}}.$$

Given the time period T of the free-particle motion and the oscillatory frequencies ω_1 and ω_2 of the two harmonic potential fields one can calculate the time intervals T_1 and T_2 from the four equations above. One sees that the inverse unitary propagator of a free particle can be realized only when the specific external harmonic potential fields are applied to the particle. Therefore, the time evolution process of a free particle is a self-irreversible evolution process, although this process is unitary.

To improve the simple method based on the free-particle motion or the inverse free-particle motion more complex multi-pulse sequences may be employed to adjust the complex linewidth. The pulse sequences consist of several pulses of the harmonic potential fields with different oscillatory frequencies. One of the

pulse sequences is given below. It is known that after the free-particle motion with a short time period T the atom is in the Gaussian wave-packet state of Eq. (53a). Now the atom is applied by a pulse sequence consisting of two harmonic potential field pulses with different oscillatory frequencies (see below) such that the time evolution process of the atom is described by the unitary propagator:

$$G(x_b, t_b, x_a, t_a) = \sqrt{\frac{m(-S_{ab})}{i2\pi\hbar}} \exp\{i(\frac{m}{2\hbar})[S_{bb}x_b^2 + S_{ab}2x_a x_b]\} \tag{54}$$

where these parameters in the propagator are given later. Now one has the initial state of Eq. (53a) with the time parameter $T = T_0$ and the propagator of Eq. (54). Then one can determine the time evolution process of the atom under the harmonic-potential-field pulse sequence. The final state of the process is given by

$$\Psi(x_b, t_b) = \exp(i\phi_0)[\frac{(\Delta x)^2}{2\pi}]^{1/4} \sqrt{\frac{2m(-S_{ab})}{i\hbar}}$$

$$\times \exp\{-\frac{1}{4}(\frac{2m}{\hbar})[(\frac{2m(\Delta x)^2}{\hbar})S_{ab}^2 + i(T_0 S_{ab}^2 - S_{bb})]x_b^2\}. \tag{55}$$

The Gaussian wave-packet state $\Psi(x_b, t_b)$ (55) has the complex linewidth:

$$W = (\frac{\hbar}{2m})\frac{\delta S_{ab}^2 - i(T_0 S_{ab}^2 - S_{bb})}{\delta^2 S_{ab}^4 + (T_0 S_{ab}^2 - S_{bb})^2} \tag{56}$$

where $\delta = 2m(\Delta x)^2/\hbar$. Denote the complex linewidth $W = (\Delta y)^2 + i\hbar T_w/(2m)$. Then one has

$$T_w = -\frac{(T_0 S_{ab}^2 - S_{bb})}{\delta^2 S_{ab}^4 + (T_0 S_{ab}^2 - S_{bb})^2} \tag{57}$$

and

$$(\Delta y)^2 = \frac{(\Delta x)^2 S_{ab}^2}{\delta^2 S_{ab}^4 + (T_0 S_{ab}^2 - S_{bb})^2}. \tag{58}$$

Obviously, $T_w \geq 0$ if $(T_0 S_{ab}^2 - S_{bb}) \leq 0$ and $T_w < 0$ if $(T_0 S_{ab}^2 - S_{bb}) > 0$. Only when $\delta^2 S_{ab}^4 + (T_0 S_{ab}^2 - S_{bb})^2 << 1$ can it be possible for the absolute parameter $|T_w|$ to be much larger than one. The equations (57) and (58) show that both the real ($(\Delta y)^2$) and imaginary (T_w) parts of the complex linewidth can be controlled simultaneously by the harmonic-potential-field pulse sequence.

Suppose that the harmonic-potential-field pulse sequence is applied to the atom in the manner that the first harmonic potential field with the oscillatory frequency ω is applied to the atom at the initial time, it lasts a time interval T, then it is turned off and at the same time the second harmonic potential field with the oscillatory frequency ω_o is turned on, and then it lasts a time interval T_o. Thus, the total time period of the harmonic-potential-field pulse sequence is $T + T_o$. It is known that the unitary propagator of a harmonic oscillator is generally written as [13, 14, 21],

$$G(x_b, t_b; x_a, t_a) = \sqrt{\frac{m\omega}{i2\pi\hbar\sin(\omega T)}} \exp\{i\frac{m\omega}{2\hbar\sin(\omega T)}[(x_b^2 + x_a^2)\cos(\omega T) - 2x_b x_a]\} \tag{59}$$

The propagator (49) is a special case of the propagator (59) with the time period $T = [2k\pi + \pi/2]/\omega$. One can see that when the time period T satisfies $\omega T = k\pi$, the propagator (59) appears singular as $\sin(\omega T) = 0$. It seems that the propagator of the harmonic oscillator should take a different form from the original one (59) at these time period points that make the propagator singular. But it can turn out that even at these time period points the correct propagator of the harmonic oscillator can be obtained directly from the original one (59) [47, 48]. Therefore, the unitary propagator (59) indeed can describe completely the time evolution process of a harmonic oscillator with any time period. The composite unitary propagator of the harmonic-potential-field pulse sequence then is calculated by

$$G(x_b, t_b; x_a, t_a) = \int dx_c G_2(x_b, t_b; x_c, t_c) G_1(x_c, t_c; x_a, t_a) \tag{60}$$

where $\{G_k(x', t'; x, t),\ k = 1,\ 2\}$ are the unitary propagators of the atom under the first ($k = 1$) and the second ($k = 2$) harmonic potential field, respectively. Note that $T = t_c - t_a$, $T_o = t_b - t_c$, and $t_b - t_a = T + T_o$. By substituting the unitary propagator (59) of the harmonic oscillator in the equation (60) and then by a complex calculation one obtains

$$G(x_b, t_b, x_a, t_a) = \sqrt{(\frac{m\omega\omega_o}{i2\pi\hbar\eta})} \exp\{i(\frac{m}{2\hbar})[S_{bb}x_b^2 + S_{ab}2x_a x_b + S_{aa}x_a^2]\} \tag{61}$$

where the parameters are given by

$$\eta = [\omega_o \cos(\omega_o T_o)\sin(\omega T) + \omega \sin(\omega_o T_o)\cos(\omega T)],$$

$$S_{bb} = \frac{\omega_o}{\eta}[-\omega_o \sin(\omega T)\sin(\omega_o T_o) + \omega \cos(\omega_o T_o)\cos(\omega T)],$$

$$S_{ab} = -\frac{\omega_o \omega}{\eta},$$

$$S_{aa} = \frac{\omega}{\eta}[-\omega \sin(\omega T)\sin(\omega_o T_o) + \omega_o \cos(\omega T)\cos(\omega_o T_o)].$$

If one sets the parameter $S_{aa} = 0$ in Eq. (61), then the propagator (61) is reduced to the propagator (54). Since the parameter η satisfies $0 < |\eta| \leq \omega + \omega_o$, the parameter $S_{aa} = 0$ means that

$$\tan(\omega T)\tan(\omega_o T_o) = \frac{\omega_o}{\omega}. \tag{62}$$

By using the equation (62) one can reduce respectively the parameters S_{bb} and S_{ab}^2 to the forms:

$$S_{bb} = \frac{(\omega^2 - \omega_o^2)}{\omega} \frac{\tan(\omega T)}{[1 + \tan^2(\omega T)]},$$

$$S_{ab}^2 = \frac{\omega_o^2 + \omega^2 \tan^2(\omega T)}{1 + \tan^2(\omega T)}.$$

From these parameters one sees that both the wave-packet spreading $\sqrt{2}(\Delta y)^2$ (58) and the time interval T_w (57) are dependent upon the parameters ω, T, ω_o, and T_o of the two harmonic potential fields. There are four undetermined parameters ω, T, ω_o, and T_o of the two harmonic potential fields, while there are only three independent equations (57), (58), and (62) to determine these parameters. Thus, given the complex linewidth $W = (\Delta y)^2 + i\hbar T_w/(2m)$ for the Gaussian wave-packet state $\Psi(x_b, t_b)$ (55), one can determine these parameters for the two harmonic potential fields.

The equations (57) and (58) lead to the relation:

$$\frac{(\Delta x)^2 T_w}{(\Delta y)^2} = -\frac{(T_0 S_{ab}^2 - S_{bb})}{S_{ab}^2}. \tag{63}$$

Substituting the parameters S_{bb} and S_{ab}^2 in the equation (63) one obtains

$$\frac{\omega T_w (\Delta x)^2}{(\Delta y)^2} = -\frac{(\omega T_0)[n_o^2 + \tan^2(\omega T)] - (1 - n_o^2)\tan(\omega T)}{[n_o^2 + \tan^2(\omega T)]} \tag{64}$$

where $n_o^2 = \omega_o^2/\omega^2$. Now the equation (64) and the parameters S_{bb} and S_{ab}^2 are used to simplify Eq. (58) to the form

$$\frac{\omega^2 (\Delta x)^2}{(\Delta y)^2}[(\frac{2m(\Delta y)^2}{\hbar})^2 + T_w^2] = \frac{[1 + \tan^2(\omega T)]}{[n_o^2 + \tan^2(\omega T)]}. \tag{65}$$

Therefore, one obtains these three independent equations (62), (64), and (65) which can be used to determine the three independent parameters n_o, $\omega_o T_o$, and ωT if one is given in advance the parameters $(\Delta y)^2$, $(\Delta x)^2$, T_w, ω, and ωT_0. First of all, one can solve Eq. (65) to obtain the parameter n_o^2 :

$$n_o^2 = \frac{1 + \{1 - (\frac{\omega^2(\Delta x)^2}{(\Delta y)^2})[(\frac{2m(\Delta y)^2}{\hbar})^2 + T_w^2]\}\tan^2(\omega T)}{(\frac{\omega^2(\Delta x)^2}{(\Delta y)^2})[(\frac{2m(\Delta y)^2}{\hbar})^2 + T_w^2]}. \tag{66}$$

Then inserting the parameter n_o^2 into Eq. (64) one obtains

$$\tan(\omega T) = -\frac{B_0}{A_0} \tag{67}$$

where the parameters A_0 and B_0 are obtained from the given parameters $(\Delta y)^2$, $(\Delta x)^2$, T_w, ω, and ωT_0 through the equations:

$$A_0 = 1 - (\frac{\omega^2(\Delta x)^2}{(\Delta y)^2})[(\frac{2m(\Delta y)^2}{\hbar})^2 + T_w^2], \tag{68a}$$

$$B_0 = \omega T_0 + \frac{\omega T_w (\Delta x)^2}{(\Delta y)^2}. \tag{68b}$$

Now the parameter ωT can be conveniently determined from Eq. (67) when the parameters A_0 and B_0 are obtained in advance. After the parameter ωT

is obtained one can determine the parameter $n_o = \omega_o/\omega$ from Eq. (66) and further obtain the parameter $\omega_o T_o$ from Eq. (62) by using the parameters ωT and n_o. Obviously, given different parameter values $(\Delta y)^2$ and T_w, while the other parameters $(\Delta x)^2$, ω, and ωT_0 are kept constant, one can obtain a different parameter set $\{\omega_o, T_o, \omega, T\}$. The set of parameters then are used to generate the two harmonic potential field pulses.

As a summary, in the above discussion one first uses a harmonic potential field to adjust the parameter $(\Delta x)^2$ and then uses a free-particle motional process and a pair of harmonic potential field pulses to adjust jointly the complex linewidth of a Gaussian wave-packet motional state. In these processes the center-of-mass position ($x_0 = 0$) and momentum ($p_0 = 0$) of the atomic Gaussian wave-packet motional state always keep unchanged due to the fact that the atomic motional momentum is zero. Finally, it can turn out that the inverse unitary propagator of a harmonic oscillator may be prepared by its unitary propagator. It is well known that the inverse propagator $U(t, t_0)^+$ of a unitary propagator $U(t, t_0)$ is just equal to the unitary propagator $U(t_0, t)$, that is, $U(t_0, t) \equiv U(t, t_0)^+$. It is known that the unitary propagator of a harmonic oscillator is given by Eq. (59). If one sets the time interval of the propagator (59) to be $T = t_b - t_a = 2k\pi/\omega - T'$ or $\omega T = 2k\pi - \omega T'$, then the propagator (59) is rewritten as

$$G(x_b, t_b; x_a, t_a) = \sqrt{-\frac{m\omega}{i2\pi\hbar \sin(\omega T')}}$$

$$\times \exp\{-i\frac{m\omega}{2\hbar \sin(\omega T')}[(x_b^2 + x_a^2)\cos(\omega T') - 2x_b x_a]\}. \tag{69}$$

By comparing Eq. (69) to Eq. (59) one sees that up to a global phase factor the unitary propagator $U(T)$ (69) is really the inverse propagator $U(T')^+$ of the harmonic oscillator with the time period T', that is, $U(T) = \exp(i\phi_0)U(T')^+$ with a global phase factor $\exp(i\phi_0)$. This means that the inverse unitary propagator of a harmonic oscillator can be generated from its unitary propagator (59). One therefore concludes that the Hamiltonian $H_0 = p^2/(2m) + m\omega^2 x^2/2$ of a harmonic oscillator can generate both the unitary propagator $U(T')$ and its inverse propagator $U(T')$ of the harmonic oscillator. This is completely different from the case of a free particle. As shown in Eq. (38) and (38a), one needs to use the inverse unitary propagator of a harmonic oscillator to build up the state-selective trigger pulse.

4. Manipulating a Gaussian wave-packet state by the unitary propagator of a general quadratic Hamiltonian

In the section it is investigated in detail how the unitary propagator of a quadratic Hamiltonian (or Lagrangian) affects an atomic Gaussian wave-packet motional state in an internal-state-independent form. A general method to manipulate a Gaussian wave-packet state of an atom is to use the unitary propagator generated by a quadratic Hamiltonian of the atom. A unitary propagator generated by a quadratic Hamiltonian does not change the Gaussian shape of

a Gaussian wave-packet state to any other shape when it acts on the Gaussian wave-packet state [5, 6, 13, 14]. Generally, a quadratic Hamiltonian of a quantum system in one dimension may be written as

$$H(t) = \frac{1}{2m}p^2 + V(x,t). \tag{70}$$

Here the generalized quadratic potential operator $V(x,t)$ consists of only linear and quadratic terms of the center-of-mass coordinate and momentum of the quantum system,

$$V(x,t) = \frac{1}{2}b(t)(px+xp) + \frac{1}{2}c(t)x^2 + d(t)p + f(t)x. \tag{71}$$

Several typical examples have been given in the previous sections. It is well known that the potential operator $V(x,t) = 0$ for a free particle, $V(x,t) = m\omega(t)^2x^2/2$ for a harmonic oscillator, and $V(x,t) = m\omega(t)^2x^2/2 + f(t)x$ for a forced harmonic oscillator. In the potential operator (71) the linear terms are only responsible for manipulating the center-of-mass position and momentum of a Gaussian wave-packet state, while the quadratic terms can be used to control the complex linewidth of a Gaussian wave-packet state. It is well known that in the coordinate representation the unitary propagator of a quadratic Hamiltonian (or Lagrangian) can be exactly obtained by the Feynman path integration [13, 14, 21]. The time evolution behavior of a quantum system with a quadratic Hamiltonian has been studied extensively and thoroughly [5, 6, 10, 11, 13, 14, 21, 47, 48, 49]. The unitary propagator of a quantum system with a quadratic Hamiltonian in one-dimensional coordinate space may be generally written as [13, 14, 21, 49]

$$G(x_b,t_b;x_a,t_a) = \sqrt{\frac{m}{i2\pi\hbar f_{ab}}}\exp\{i\frac{m}{2\hbar}[S_{bb}x_b^2 + S_{ab}2x_ax_b + S_{aa}x_a^2]\}$$

$$\times\exp\{+\frac{i}{\hbar}x_aQ_a(t_b,t_a) + \frac{i}{\hbar}x_bQ_b(t_b,t_a)\}\exp[i\Theta(t_b,t_a)] \tag{72}$$

where the function $f_{ab} = (-S_{ab})^{-1}$ [49]. Some important and frequently using unitary propagators which are the special forms of Eq. (72) have been given in the previous sections: (i) the Hamiltonian of a free particle is $H = p^2/(2m)$ and the unitary propagator is given by Eq. (52a); (ii) the Hamiltonian of a harmonic oscillator is given by H_0 of Eq. (2) and the unitary propagator is given by Eq. (59); (iii) a forced harmonic oscillator has the Hamiltonian of Eq. (3) and its unitary propagator is given by [13, 14, 21]

$$G_f(x_b,t_b;x_a,t_a) = \exp\{i\Theta(t_b,t_a)\}G(x_b,t_b;x_a,t_a)$$

$$\times\exp\{\frac{i}{\hbar}[Q_b(t_b,t_a)x_b + Q_a(t_b,t_a)x_a]\} \tag{73}$$

where $G(x_b,t_b;x_a,t_a)$ is given by Eq. (59) with the time period $T=t_b-t_a$ and

$$Q_a(t_b,t_a)=-\frac{1}{\sin(\omega T)}\int_{t_a}^{t_b}f(t)\sin[\omega(t_b-t)]dt,$$

$$Q_b(t_b,t_a)=-\frac{1}{\sin(\omega T)}\int_{t_a}^{t_b}f(t)\sin[\omega(t-t_a)]dt,$$

$$\Theta(t_b,t_a)=-\frac{1}{m\omega\hbar\sin(\omega T)}\int_{t_a}^{t_b}\int_{t_a}^{t}f(t)f(s)\sin[\omega(t_b-t)]\sin[\omega(s-t_a)]dsdt.$$

It has been shown that the unitary propagator (73) of a forced harmonic oscillator can be used to generate a standard coherent state of a harmonic oscillator [13, 14, 15, 16, 17]. It is known in the section 2 that the harmonic potential field $(m\omega(t)^2x^2/2)$ and the forced field $(f(t)x)$ for a harmonic oscillator can be generated by the external driving electric or magnetic field.

If one names the unitary propagator (72) of a quadratic Hamiltonian the quadratic unitary propagator, then it can turn out that a product of any two quadratic unitary propagators is still a quadratic unitary propagator. This property is particularly important as it leads to that a complex quadratic unitary propagator may be decomposed into a sequence of simple quadratic unitary propagators. This makes it convenient to implement a complex quadratic unitary propagator in experiment, since a simple quadratic unitary propagator can be prepared easily in experiment. The unitary propagator (72) consists of the quadratic terms x_b^2, $2x_ax_b$, and x_a^2 and the linear terms x_b and x_a in addition to the global phase $\Theta(t_b,t_a)$. Suppose that there are two quadratic unitary propagators, each of which contains only quadratic terms x_b^2, $2x_ax_b$, and x_a^2. Then it can prove that a product of the two unitary propagators is still a quadratic unitary propagator that contains only the quadratic terms. This is a direct result of the Lie group generated by the Lie algebra $su(1,1)$ whose three basis elements may be taken as p^2, x^2, and $(px+xp)/2$. On the other hand, a quadratic unitary propagator which contains linear terms times another quadratic unitary propagator that contains only quadratic terms will generate a quadratic unitary propagator that contains linear terms. The quadratic terms of the unitary propagator (72) can manipulate the complex linewidth of a Gaussian wave-packet state, while the linear terms are used to control only the center-of-mass position and momentum of the Gaussian wave-packet state. This will be proven below.

It is known that a standard Gaussian wave-packet state of an atom with mass m may be written as [2, 3, 4, 5, 6, 7, 8]

$$\Psi_0(x,t)=\exp(i\phi_0)[\frac{(\Delta x)^2}{2\pi}]^{1/4}\sqrt{\frac{1}{[(\Delta x)^2+i(\frac{\hbar T_0}{2m})]}}$$

$$\times\exp\{-\frac{1}{4}\frac{(x-x_0)^2}{[(\Delta x)^2+i(\frac{\hbar T_0}{2m})]}\}\exp[-ip_0x/\hbar] \qquad (74)$$

where x_0 and $-p_0$ are the center-of-mass position and momentum of the Gaussian wave-packet state, respectively, $\exp(i\phi_0)$ is a global phase factor, and the complex linewidth of the Gaussian wave-packet state is given by

$$W = (\Delta x)^2 + i(\frac{\hbar T_0}{2m}).$$

The Gaussian wave-packet state $\Psi_0(x,t)$ (74) has the wave-packet spreading $\varepsilon(T_0) = \sqrt{2[(\Delta x)^2 + (\frac{\hbar T_0}{2m(\Delta x)})^2]}$. The relation between the wave-packet spreading and the complex linewidth is given by

$$|W|^2 = \frac{1}{2}(\Delta x)^2 \varepsilon(T_0)^2. \tag{75}$$

The physical meaning for the Gaussian wave-packet state $\Psi_0(x,t)$ is clear: if a free atom is in the Gaussian wave-packet state, then the Gaussian wave-packet state tells ones that the atom moves along the direction x in the coordinate axis with the motional velocity $(-p_0/m)$. A Gaussian wave-packet state is completely described by the three parameters: the center-of-mass position x_0, the mean momentum $(-p_0)$, and the complex linewidth W. Now examine the time evolution process of the Gaussian wave-packet state $\Psi_0(x_a,t_a)$ (74) under the action of the unitary propagator $G(x_b,t_b;x_a,t_a)$ (72) of a quadratic Hamiltonian. The time evolution process may be calculated by

$$\Psi(x_b,t_b) = \int dx_a G(x_b,t_b;x_a,t_a)\Psi_0(x_a,t_a)$$

$$= \exp(i\phi_0)[\frac{(\Delta x)^2}{2\pi}]^{1/4}\sqrt{\frac{1}{[(\Delta x)^2 + i(\frac{\hbar T_0}{2m})]}}\sqrt{\frac{m}{i2\pi\hbar f_{ab}}}$$

$$\times \int_{-\infty}^{\infty} dx_a \{\exp\{iS_c/\hbar\}\exp\{-\frac{1}{4}\frac{(x_a - x_0)^2}{[(\Delta x)^2 + i(\frac{\hbar T_0}{2m})]}\}\exp\{-ip_0 x_a/\hbar\}\} \tag{76a}$$

where the action S_c can be found from the propagator (72),

$$S_c = \frac{1}{2}m[S_{bb}x_b^2 + S_{ab}2x_a x_b + S_{aa}x_a^2] + x_a Q_a(t_b,t_a) + x_b Q_b(t_b,t_a) + \hbar\Theta(t_b,t_a).$$

One can write the final state $\Psi(x_b,t_b)$ as, with the help of the Gaussian integral (50),

$$\Psi(x_b,t_b) = \exp(i\phi_0)\exp[i\Theta(t_b,t_a)][\frac{(\Delta x)^2}{2\pi}]^{1/4}\sqrt{\frac{1}{[(\Delta x)^2 + i(\frac{\hbar T_0}{2m})]}}\sqrt{\frac{m}{i2\pi\hbar f_{ab}}}$$

$$\times \exp\{-\frac{1}{4}\frac{x_0^2}{[(\Delta x)^2 + i(\frac{\hbar T_0}{2m})]}\}\exp\{\frac{im}{2\hbar}S_{bb}x_b^2\}\exp\{\frac{i}{\hbar}x_b Q_b(t_b,t_a)\}\sqrt{\frac{\pi}{a}}\exp(\frac{b^2}{4a}), \tag{76b}$$

where the parameters a and b are given by

$$a = -i\frac{m}{2\hbar}S_{aa} + \frac{1}{4}\frac{1}{[(\Delta x)^2 + i(\frac{\hbar T_0}{2m})]} \tag{77a}$$

and

$$b = \frac{im}{\hbar}S_{ab}x_b + \frac{i}{\hbar}[Q_a(t_b,t_a) - p_0] + \frac{1}{2}\frac{x_0}{[(\Delta x)^2 + i(\frac{\hbar T_0}{2m})]}. \tag{77b}$$

Since $b^2/(4a)$ is quadratic with the coordinate x_b, the wave function $\Psi(x_b,t_b)$ of Eq. (76b) can be written as

$$\Psi(x_b,t_b) = \exp(i\phi_0)\exp[i\Theta(t_b,t_a)][\frac{(\Delta x)^2}{2\pi}]^{1/4}$$

$$\times\sqrt{\frac{1}{f_{ab}}\frac{1}{S_{aa}(\Delta x)^2 + i(\frac{\hbar}{2m})[1+T_0S_{aa}]}}\exp\{Ax_b^2 + Bx_b + C)\} \tag{78}$$

where the complex coefficients A, B, and C are determined through

$$Ax_b^2 + Bx_b + C = \frac{b^2}{4a} + \frac{im}{2\hbar}S_{bb}x_b^2 + \frac{i}{\hbar}x_bQ_b(t_b,t_a) - \frac{1}{4}\frac{x_0^2}{[(\Delta x)^2 + i\frac{\hbar T_0}{2m}]}. \tag{79}$$

By substituting the parameter a of Eq. (77a) and b of Eq. (77b) in the equation (79) one can obtain explicitly the complex coefficients A, B, and C. Furthermore one can find, by a complex calculation,

$$\exp\{\mathrm{Re}(A)x_b^2 + \mathrm{Re}(B)x_b + \mathrm{Re}(C)\}$$

$$= \exp\{-\frac{1}{4}S_{ab}^2\frac{\{x_b + S_{ab}^{-1}[S_{aa}x_0 + (\frac{Q_a(t_b,t_a)-p_0}{m})]\}^2}{S_{aa}^2(\Delta x)^2 + (\frac{\hbar}{2m(\Delta x)})^2[1+S_{aa}T_0]^2}\}.$$

Therefore, the wave function $\Psi(x_b,t_b)$ of Eq. (78) can be further written as

$$\Psi(x_b,t_b) = \exp(i\phi_0)\exp[i\Theta(t_b,t_a)]\exp\{i\,\mathrm{Im}(C)\}$$

$$\times[\frac{(\Delta x)^2}{2\pi}]^{1/4}\sqrt{\frac{(-S_{ab})}{S_{aa}(\Delta x)^2 + i(\frac{\hbar}{2m})[1+T_0S_{aa}]}}$$

$$\times\exp\{-\frac{1}{4}S_{ab}^2\frac{\{x_b + S_{ab}^{-1}[S_{aa}x_0 + (\frac{Q_a(t_b,t_a)-p_0}{m})]\}^2}{S_{aa}^2(\Delta x)^2 + (\frac{\hbar}{2m(\Delta x)})^2[1+S_{aa}T_0]^2}\}$$

$$\times\exp\{i[\mathrm{Im}(A)x_b^2 + \mathrm{Im}(B)x_b]\}. \tag{80}$$

The wave function $\Psi(x_b,t_b)$ of Eq. (80) is indeed a Gaussian wave-packet state. This can be seen more clearly from its probability density $|\Psi(x_b,t_b)|^2$ which is

a standard Gaussian function. Thus, the Gaussian wave-packet state $\Psi(x_b, t_b)$ (80) has the center-of-mass position:

$$x_c = -S_{ab}^{-1}[S_{aa}x_0 + (\frac{Q_a(t_b,t_a) - p_0}{m})]$$

and the wave-packet spreading:

$$\varepsilon = \sqrt{2S_{ab}^{-2}\{S_{aa}^2(\Delta x)^2 + (\frac{\hbar}{2m(\Delta x)})^2[1 + T_0 S_{aa}]^2\}}.$$

The imaginary parts of the coefficients A, B, and C in the state $\Psi(x_b, t_b)$ (80) are given by

$$\mathrm{Im}(A) = (\frac{m}{2\hbar})\frac{S_{bb}(1 + S_{aa}T_0) + Z(T_0)(S_{bb}S_{aa} - S_{ab}^2)}{\delta^2 S_{aa}^2 + [1 + S_{aa}T_0]^2}, \tag{81a}$$

$$\mathrm{Im}(B) = (\frac{m}{\hbar})\frac{(v_b + S_{ab}x_0)(1 + S_{aa}T_0) + Z(T_0)(v_b S_{aa} - v_a S_{ab})}{[1 + S_{aa}T_0]^2 + \delta^2 S_{aa}^2}, \tag{81b}$$

$$\mathrm{Im}(C) = (\frac{m}{2\hbar})\frac{(x_0^2 S_{aa} + 2x_0 v_a - T_0 v_a^2)[1 + S_{aa}T_0] - \delta^2 S_{aa} v_a^2}{\delta^2 S_{aa}^2 + [1 + S_{aa}T_0]^2}, \tag{81c}$$

where $Z(T_0) = [S_{aa}\delta^2 + T_0(1 + S_{aa}T_0)]$ and the parameters δ, v_a, and v_b are defined by

$$\delta = (\frac{2m(\Delta x)^2}{\hbar}), \quad v_a = \frac{1}{M}[Q_a(t_b,t_a) - p_0], \quad v_b = \frac{1}{M}Q_b(t_b,t_a).$$

The state $\Psi(x_b, t_b)$ (80) can be further written as a standard Gaussian wave-packet state:

$$\Psi(x,t) = \exp(i\varphi_0)[\frac{(\Delta y)^2}{2\pi}]^{1/4}\sqrt{\frac{1}{[(\Delta y)^2 + i(\frac{\hbar T_w}{2m})]}}$$

$$\times \exp\{-\frac{1}{4}\frac{(x - x_c)^2}{[(\Delta y)^2 + i(\frac{\hbar T_w}{2m})]}\}\exp[-ipx/\hbar] \tag{82}$$

where $\exp(i\varphi_0)$ is a global phase factor, the mean momentum $(-p)$ is given through

$$p/\hbar = -\,\mathrm{Im}(B) - 2x_c\,\mathrm{Im}(A), \tag{83}$$

the real part of the complex linewidth is

$$(\Delta y)^2 = \frac{S_{ab}^2\{S_{aa}^2(\Delta x)^2 + (\frac{\hbar}{2m(\Delta x)})^2[1 + S_{aa}T_0]^2\}}{S_{ab}^4 + 16(\mathrm{Im}(A))^2\{S_{aa}^2(\Delta x)^2 + (\frac{\hbar}{2m(\Delta x)})^2[1 + S_{aa}T_0]^2\}^2}, \tag{84}$$

and the imaginary part equals

$$(\frac{\hbar T_w}{2m}) = \frac{4\,\mathrm{Im}(A)\{S_{aa}^2(\Delta x)^2 + (\frac{\hbar}{2m(\Delta x)})^2[1+S_{aa}T_0]^2\}^2}{S_{ab}^4 + 16(\mathrm{Im}(A))^2\{S_{aa}^2(\Delta x)^2 + (\frac{\hbar}{2m(\Delta x)})^2[1+S_{aa}T_0]^2\}^2}. \qquad (85)$$

Here the imaginary part of the complex linewidth can be positive or negative, which is dependent on the term $\mathrm{Im}(A)$. It is known from Eq. (81a) that the term $\mathrm{Im}(A)$ is dependent on only the parameters S_{bb}, S_{ab}, and S_{aa} of the quadratic terms in the unitary propagator (72) of the quadratic Hamiltonian (70) but independent of those parameters of the linear terms. Then the real $((\Delta y)^2)$ and the imaginary (T_w) part of the complex linewidth are dependent on only the quadratic terms but not the linear terms of the unitary propagator (72). One therefore concludes that the complex linewidth of a Gaussian wave-packet state can be adjusted only by the quadratic terms in the unitary propagator (72). This is a convenient method to manipulate the complex linewidth of a Gaussian wave-packet state by adjusting only the quadratic terms of the unitary propagator (72). In the quadratic Hamiltonian (70) the linear terms $d(t)p$ and $f(t)x$ do not have a contribution to the quadratic terms of the unitary propagator (72) [13, 14, 21, 49]. Then one can use only the quadratic operator terms of the Hamiltonian (70) to adjust the quadratic terms of the unitary propagator (72). Consequently one can manipulate at will the complex linewidth of a Gaussian wave-packet state by the quadratic operator terms of the quadratic Hamiltonian (70). On the other hand, it is known from Eq. (81b) that the term $\mathrm{Im}(B)$ is dependent on the parameters $Q_a(t_b, t_a)$ and $Q_b(t_b, t_a)$ of the linear terms of the unitary propagator (72). Then the center-of-mass position x_c and momentum $(-p)$ of the Gaussian wave-packet state (82) are dependent on the parameters $Q_a(t_b, t_a)$ and/or $Q_b(t_b, t_a)$, although they are also dependent on those parameters of the quadratic terms. One therefore can manipulate the center-of-mass position and momentum of a Gaussian wave-packet state through the linear terms of the unitary propagator (72) or more conveniently through the linear terms of the quadratic Hamiltonian (70). A typical example is that in order to generate a standard coherent state of harmonic oscillator with a higher motional energy one may use the linear term $f(t)x$ [13, 14, 15, 16, 17, 21, 49] that may be generated by the external driving field, while one may use the time-dependent and frequency-varying harmonic potential field to adjust the complex linewidth of a Gaussian wave-packet state, as shown in the previous section 3.

5. Discussion

The state-selective trigger pulse has the two basic properties. One of which is that the state-selective trigger pulse can have a real effect on the atom only when the atom is in some given internal states. Another is that the state-selective trigger pulse does not change Gaussian shape of an atomic Gaussian wave-packet motional state to any other shape. The former property is inherent and the last one is due to the fact that a Gaussian wave-packet motional state for a single atom is simple and easy to be manipulated and controlled in time and space. An internal-state-dependent selective excitation process of an atomic system is

generally closely related to manipulation and control in time and space of the atomic internal electronic (or spin) motion, the atomic center-of-mass motion, and the coupling between the internal and the center-of-mass motion, although an inhomogeneous external magnetic field could also generate an internal-state-dependent force exerted on a spin and hence could be used to generate in an internal-state selective form a coherent state of the spin in the harmonic potential field. Therefore, a general construction for the state-selective trigger pulse is generally involved in using the electromagnetic field pulses, i.e., the laser light pulses, to create the interaction between the atomic internal states and center-of-mass motional states and realize the coupling between the center-of-mass and the internal motion of the atom. A state-selective trigger pulse transfers one atomic Gaussian wave-packet motional state to another with the help of the atomic internal states. This point is the same as those of the unitary decelerating and accelerating processes [8]. On the other hand, there are many other space-dependent processes in quantum information science. Typical examples include the quantum communication [52] and the construction of quantum gates by the short-range interactions such as the conditional collision interaction [53]. These space-dependent processes generally use the particle (photon or atom) transport process to realize the quantum state transfer of the atomic internal states or the photon polarization states in space and implement the quantum gate operations of the internal-state quantum bits. Here motional states are usually considered as carrier of the quantum information transport. These processes generally emphasize realization of the internal-state transfer in space (i.e., quantum information transfer) or the quantum gate operations of the atomic internal-state quantum bits with the help of the motional states of the particles instead of the unitary manipulation of the motional states themselves. In this sense these space-dependent processes are different from the state-selective excitation process of the trigger pulse and the unitary decelerating and accelerating processes. In the reversible and unitary halting protocol [1] the quantum program converts the difference of the atomic internal states (the initial functional states) into the difference between the wave-packet motional states of the halting-qubit atom in space. Thus, one has to manipulate in time and space the atomic wave-packet motional states, in order that the reversible and unitary halting protocol is state-insensitive. Note that here the atomic wave-packet motional states are not used as quantum bits and the atomic internal states still act as the halting quantum bit.

Manipulating unitarily in space a wave-packet motional state of an atom or a superposition of motional states of the atom is generally difficult in experiment with respect to manipulating a purely time-dependent quantum state. However, it is of crucial importance to realize both the reversible and unitary state-insensitive halting protocol and the efficient quantum search process. The quantum control process that simulates the reversible and unitary halting protocol contains the conventional particle transport process such as the free-particle motion, but the more important is that it also contains the coherent-state selective excitation process of the trigger pulse and the unitary decelerating and accelerating processes that are different from the conventional particle transport

process. The conventional transport process usually need not require the cooperation of the center-of-mass and the internal motion of the atom, while these unitary processes need to manipulate not only the atomic internal and center-of-mass motions in space but also their coupling. Unitary manipulation for the atomic internal states which act as quantum bits is an important research area in quantum computation in the past decade, and it is usually more convenient than for atomic wave-packet motional states. One of the main reasons why it is generally difficult to manipulate at will an atomic wave-packet motional state in space is that the unitary dynamical process is generally complicated for an atom in a general potential field. In few simplest potential fields such as a harmonic potential field the quantum dynamical behavior of an atom can be completely understood, while a complete knowledge for an atom system and its quantum dynamics is the basis to manipulate at will the atomic motional states in space. Thus, at present the unitary manipulation in time and space of motional states of the halting-qubit atom in the quantum control process has to be restricted to a simplest case such as the atom in a Gaussian wave-packet motional state and in a harmonic potential field. In contrast to the preparation of quantum gate operations, here the unitary manipulation in time and space of the atomic motional states becomes the main research area.

The quantum dynamics of a harmonic oscillator and a Gaussian wave-packet state have been studied extensively and thoroughly in quantum mechanics. A Gaussian wave-packet state is one of the simplest quantum wave-packet states that can be manipulated and controlled in time and space easily and precisely. In the previous paper [8] it has been shown that the unitary decelerating and accelerating processes based on the STIRAP method can transfer one Gaussian wave-packet motional state of a free atom to another in the ideal or near ideal adiabatic condition and can manipulate the center-of-mass position and momentum of a Gaussian wave-packet motional state. The advantage of the manipulation is that the space-selective manipulation can be carried out easily, since the manipulation uses the STIRAP laser light pulse sequence, while laser light makes it easy to perform the space-selective and internal-state-selective operations of an atom. However, the shortcoming of the manipulation is that the complex linewidth of a Gaussian wave-packet motional state can not be manipulated at will by the STIRAP method. Now in this paper several methods have been developed to manipulate the complex linewidth of a Gaussian wave-packet motional state of an atom. Their basic starting point is to apply a quadratic potential field to the atom. These results in the previous [8] and the present paper show that a Gaussian wave-packet motional state of an atom can be manipulated at will in experiment. It can be predicted from these results that there is no longer unsurpassable obstacle in theory for a quantum computer to solve the unsorted quantum search problem in polynomial time.

References

1. X. Miao, *The basic principles to construct a generalized state-locking pulse field and simulate efficiently the reversible and unitary halting protocol of a universal quantum computer*, http://arxiv.org/abs/quant-ph/0607144 (2006)

2. L. I. Schiff, *Quantum mechanics*, 3rd, McGraw-Hill book company, New York, 1968
3. M. L. Goldberger and K. M. Watson, *Collision theory*, Chapt. 3, Wiley, New York, 1964
4. R. G. Newton, *Scattering theory of waves and particles*, Chapt. 6, McGraw-Hill, New York, 1966.
5. E. J. Heller, *Time-dependent approach to semiclassical dynamics*, J. Chem. Phys. 62, 1544 (1975)
6. D. Huber and E. J. Heller, *Generalized Gaussian wave packet dynamics*, J. Chem. Phys. 87, 5302 (1987)
7. R. G. Littlejohn, *The semiclassical evolution of wave packets*, Phys. Rep. 138, 193 (1986)
8. X. Miao, *The STIRAP-based unitary decelerating and accelerating processes of a single free atom*, http://arxiv.org/abs/quant-ph/0707.0063 (2007)
9. K. Bergmann, H. Theuer, and B. W. Shore, *Coherent population transfer among quantum states of atoms and molecules*, Rev. Mod. Phys. 70, 1003 (1998)
10. R. J. Glauber, *Coherent and incoherent states of the radiation field*, Phys. Rev. 131, 2766 (1963)
11. (a) R. F. Bishop and A. Vourdas, *Generalised coherent states and Bogoliubov transformations*, J. Phys. A 19, 2525 (1986); (b) J. N. Hollenhorst, *Quantum limits on resonant-mass gravitational-radiation detectors*, Phys. Rev. D 19, 1669 (1979)
12. X. Miao, *Quantum search processes in the cyclic group state spaces*, http://arxiv.org/abs/quant-ph/0507236 (2005)
13. L. S. Schulman, *Techniques and applications of path integration*, Dover, New York, 2005
14. R. P. Feynman, *Space-time approach to non-relativistic quantum mechanics*, Rev. Mod. Phys. 20, 367 (1948); R. P. Feynmann and A. R. Hibbs, *Quantum mechanics and path integrals*, McGraw-Hill, New York, 1965
15. K. Husimi, *Miscellanea in elementary quantum mechanics II*, Prog. Theor. Phys. 9, 381 (1953)
16. E. H. Kerner, *Note on the forced and damped oscillator in quantum mechanics*, Can. J. Phys. 36, 371 (1957)
17. P. Carruthers and M. M. Nieto, *Coherent states and the forced quantum oscillator*, Am. J. Phys. 33, 537 (1965)
18. (a) D. M. Meekhof, C. Monroe, B. E. King, W. M. Itano, and D. J. Wineland, *Generation of nonclassical motional states of a trapped atom*, Phys. Rev. Lett. 76, 1796 (1996); Erratum, 77, 2346 (1996); (b) C. Monroe, D. M. Meekhof, B. E. King, and D. J. Wineland, *A "Schrödinger cat" superposition state of an atom,* Science 272, 1131 (1996)
19. D. J. Wineland, C. Monroe, W. M. Itano, D. Leibfried, B. E. King, D. M. Meekhof, *Experimental issues in coherent quantum-state manipulation of trapped atomic ions*, J. Res. NIST, 103, 259 (1998)
20. D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, *Quantum dynamics of single trapped ion*, Rev. Mod. Phys. 75, 281 (2003)

21. C. Grosche and F. Steiner, *Handbook of Feynman path integrals*, Springer, Berlin, 1998
22. (a) S. Chu, *Nobel Lecture: The manipulation of neutral particles*, Rev. Mod. Phys. 70, 685 (1998)

(b) C. N. Cohen-Tannoudji, *Nobel Lecture: Manipulating atoms with photons*, Rev. Mod. Phys. 70, 707 (1998)

(c) W. D. Phillips, *Nobel Lecture: Laser cooling and trapping of neutral atoms*, Rev. Mod. Phys. 70, 721 (1998)
23. J. I. Cirac and P. Zoller, *Quantum computations with cold trapped ions*, Phys. Rev. Lett. 74, 4091 (1995)
24. C. A. Blockley, D. F. Walls, and H. Risken, *Quantum collapses and revivals in a quantized trap*, Europhys. Lett. 17, 509 (1992)
25. (a) J. I. Cirac, A. S. Parkins, R. Blatt, and P. Zoller, *"Dark" squeezed states of the motion of a trapped ion*, Phys. Rev. Lett. 70, 556 (1993); (b) J. I. Cirac, R. Blatt, and P. Zoller, *Nonclassical states of motion in a three-dimensional ion trap by addiabatic passage*, Phys. Rev. A 49, R3174 (1994)
26. W. Vogel and R. L. de Matos Filho, *Nonlinear Jaynes-Cummings dynamics of a trapped ion*, Phys. Rev. A 52, 4214 (1995);
27. C. Monroe, D. M. Meekhof, B. E. King, S. R. Jefferts, W. M. Itano, D. J. Wineland, and P. Gould, *Ressolved-sideband Raman cooling of a bound atom to the 3D zero-point energy*, Phys. Rev. Lett. 75, 4011 (1995)
28. P. Marte, P. Zoller, and J. L. Hall, *Coherent atomic mirrors and beam splitters by adiabatic passage in multilevel systems*, Phys. Rev. A 44, R4118 (1991)
29. M. Weitz, B. C. Young, and S. Chu, *Atom manipulation based on delayed laser pulses in three- and four-level systems: light shifts and transfer efficiencies*, Phys. Rev. A 50, 2438 (1994)
30. J. L. Sørensen, D. Moller, T. Iversen, J. B. Thomsen, F. Jensen, P. Staanum, D. Voigt, and M. Drewsen, *Efficient coherent internal state transfer in trapped ions using stimulated Raman adiabatic passage*, http://arxiv.org/abs/quant-ph /0608089 (2006)
31. S. Wallentowitz and W. Vogel, *Quantum-mechanical counterpart of nonlinear optics*, Phys. Rev. A 55, 4438 (1997)
32. G. S. Agarwal and J. Banerji, *Quantum evolution of classical nonlinear eigenmode in parametric interaction and realization in traps with ions*, Phys. Rev. A 55, R4007 (1997)
33. J. Steinbach, J. Twamley, and P. L. Knight, *Engineering two-mode interactions in ion traps*, Phys. Rev. A 56, 4815 (1997)
34. L. Allen and J. H. Eberly, *Optical resonance and two-level atoms*, Dover, New York, 1987
35. E. T. Jaynes and F. W. Cummings, *Comparison of quantum and semiclassical radiation theories with application to the beam maser*, Proc. Inst. Electr. Eng. 51, 89 (1963)
36. See, for example, (a) J. Oreg, F. T. Hioe, and J. H. Eberly, *Adiabatic following in multilevel systems*, Phys. Rev. A 29, 690 (1984); (b) E. Brion, L. H. Pedersen, and K. Molmer, *Adiabatic elimination in a Lambda system*, http://arxiv.org/abs/quant-ph/0610056 (2006)

37. M. Matti Maricq, *Application of average Hamiltonian theory to the NMR of solids*, Phys. Rev. B 25, 6622 (1982)
38. C. K. Law and J. H. Eberly, *Arbitrary control of a quantum electromagnetic field*, Phys. Rev. Lett. 76, 1055 (1996)
39. W. Magnus, *On the exponential solution of differential equations for a linear operator*, Commun. Pure Appl. Math. 7, 649 (1954)
40. H. F. Trotter, *On the product of semigroups of operators*, Proc. Am. Math. Soc. 10, 545 (1959)
41. R. M. Wilcox, *Exponential operators and parameter differentiation in quantum physics*, J. Math. Phys. 8, 962 (1967)
42. M. Suzuki, *Fractal decomposition of exponential operators with applications to many-body theories and Monte Carlo simulations*, Phys. Lett. A 146, 319 (1990); *General theory of higher-order decomposition of exponential operators and symplectic integrators*, Phys. Lett. A 165, 387 (1992)
43. (a) H. Yoshida, *Construction of higher order symplectic integrators*, Phys. Lett. A 150, 262 (1990); (b) W. S. Zhu and X. S. Zhao, *Numerical quantum propagation with time-dependent Hamiltonian*, J. Chem. Phys. 105, 9536 (1996)
44. A. Sϕrensen and K. Mϕlmer, *Quantum computation with ions in thermal motion*, Phys. Rev. Lett. 82, 1971 (1999)
45. J. Milburn, *Simulating nonlinear spin models in an ion trap*, http://arxiv.org /abs/quant-ph/9908037 (1999)
46. J. F. Poyatos, J. I. Cirac, and P. Zoller, *Quantum gates with "hot" trapped ions*, Phys. Rev. Lett. 81, 1322 (1998)
47. P. A. Horvathy, *Extended Feynman formula for harmonic oscillator*, Int. J. Theor. Phys. 18, 245 (1979)
48. I. H. Duru, *Feynman′s formula for a harmonic oscillator*, Int. J. Theor. Phys. 23, 567 (1984)
49. (a) M. M. Mizrahi, *Phase space path integrals, without limiting procedures*, J. Math. Phys. 19, 298 (1978); (b) D. C. Khandekar and S. V. Lawande, *Exact solution of a time-dependent quantal harmonic oscillator with damping and a perturbative force*, J. Math. Phys. 20, 1870 (1979); (c) B. K. Cheng, *The propagator of the time-dependent forced harmonic oscillator with time-dependent damping*, J. Math. Phys. 27, 217 (1986); (d) J. T. Marshall and J. T. Pell, *Path-integral evaluation of the space-time propagator for quadratic Hamiltonian systems*, J. Math. Phys. 20, 1297 (1979)
50. A. Abragam, *Principles of nuclear magnetism*, Oxford University Press, London, 1961
51. X. Miao, *Multiple-quantum operator algebra spaces and description for the unitary time evolution of multilevel spin systems*, Molec. Phys. 98, 625 (2000)
52. C. H. Bennett and G. Brassard, *Quantum cryptography : public key distribution and coin tossing, Proceedings of IEEE international conference on computers, systems, and signal processing,* pp. 175–179, IEEE, New York, 1984
53. T. Calarco, E. A. Hinds, D. Jaksch, J. Schmiedmayer, J. I. Cirac, and P. Zoller, *Quantum gates with neutral atoms: Controlling collisional interactions in time dependent traps*, Phys. Rev. A 61, 022304 (2000)
54. (a) U. Haeberlen and J. Waugh, *Coherent averaging effects in magnetic*

resonance, Phys. Rev. 175, 453 (1968); (b) U. Haeberlen, *High resolution NMR in solids*, Adv. Magn. Reson. Suppl. 1, 1976